Solving a Challenging Quadratic 3D Assignment Problem

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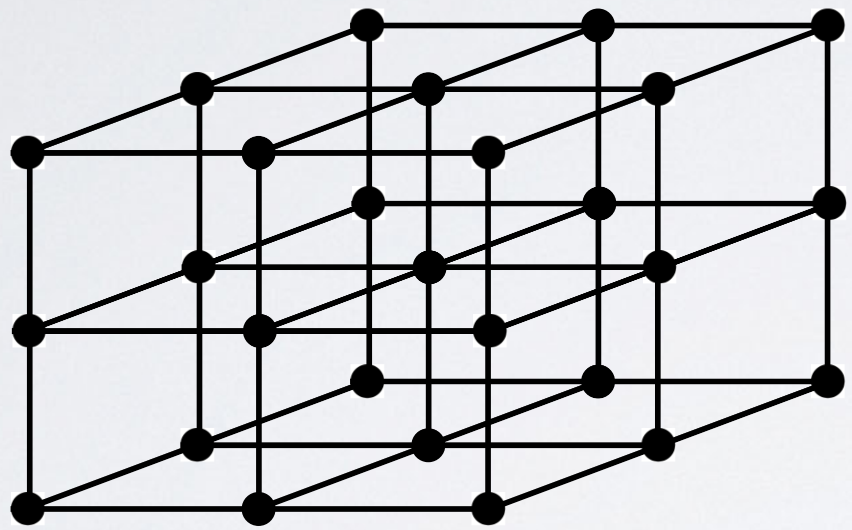
Quadratic 3D Assignment Problem



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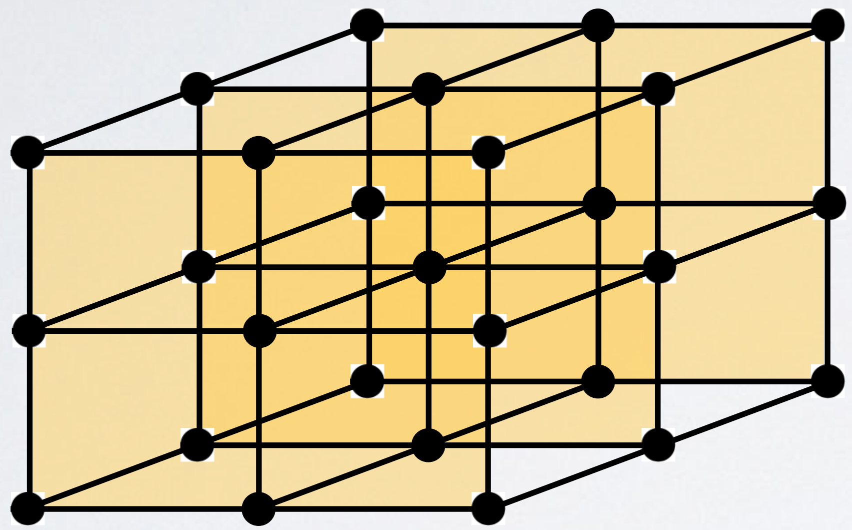

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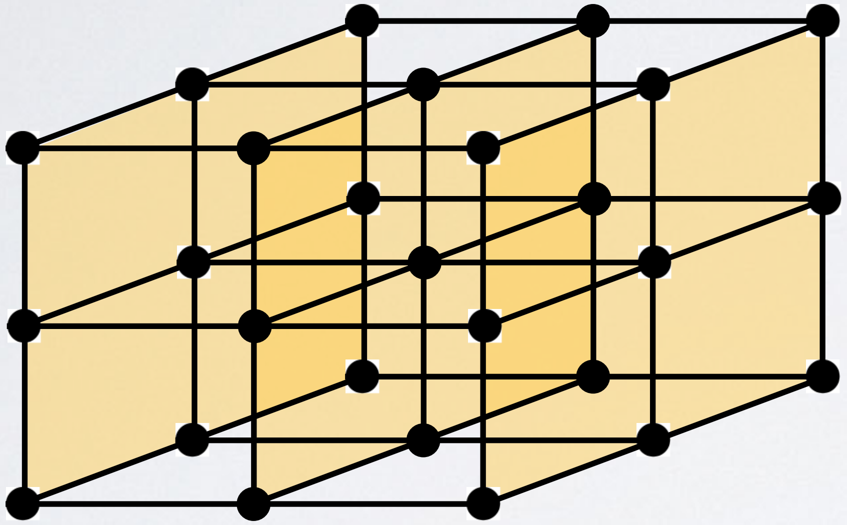

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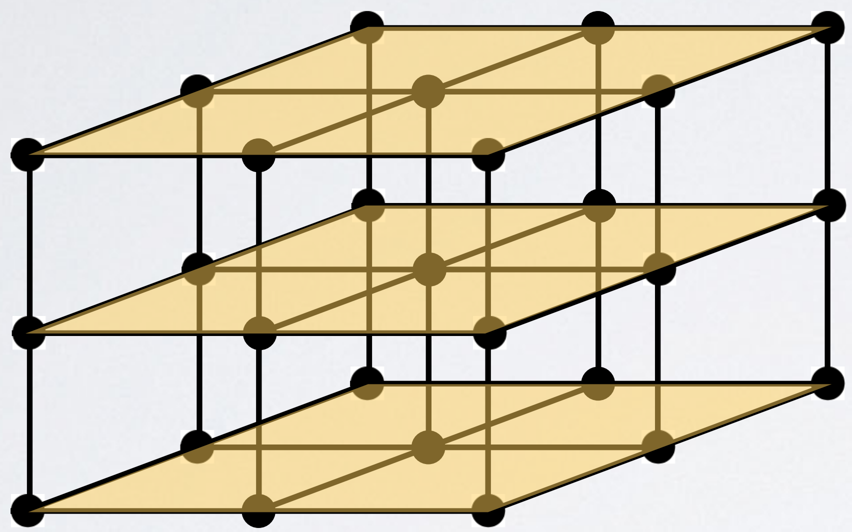

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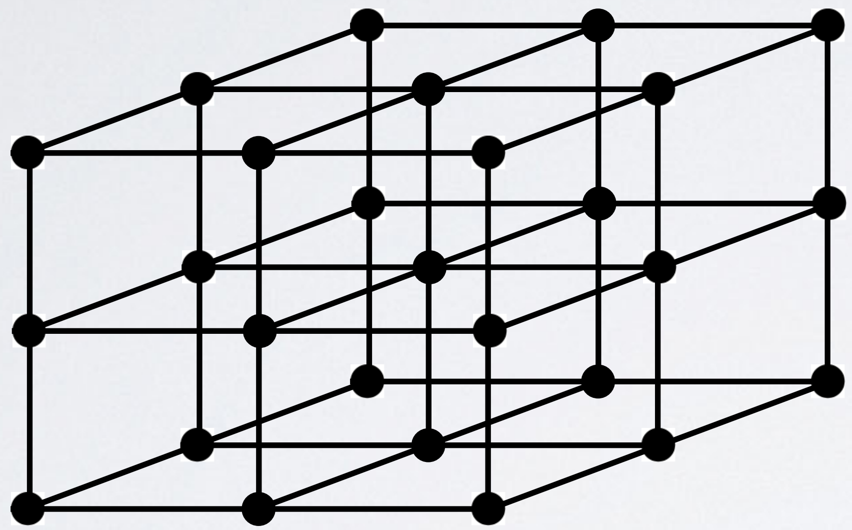

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quadratic objective $c_{ijkpqr} x_{ijk} x_{pqr}$

Our Instance

16-PSK digital communication retransmission protocol

$$c_{\boxed{i}jk\boxed{p}qr}$$

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- high dynamism 3.6×10^{12}
- symmetric: group of order 49.152

```
Parallel mode: deterministic, using up to 16 threads.
Root relaxation solution time = 43.45 sec. (17943.71 ticks)

        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

*     0+    0                      207392.0000        0.0000     4538  100.00%
      0     0     9832.0000    50   207392.0000     9832.0000     4538   95.26%
      0     0     9832.0000    50   207392.0000      Cuts: 12     5450   95.26%
      0     0     9832.0000    55   207392.0000      Cuts: 25     5569   95.26%
Heuristic still looking.
      0     2     9832.0000    51   207392.0000     9832.0000     5569   95.26%
Elapsed time = 186.17 sec. (166732.53 ticks, tree = 0.01 MB, solutions = 1)
      1     3    10639.0000    39   207392.0000     9832.0000    11529   95.26%
…
…
…
 25953439 25481647    16202.0920    34   207392.0000    11819.7613 3.89e+08   94.30%
 25958606 25486740        cutoff         207392.0000    11819.8664 3.89e+08   94.30%
 25965184 25493181    27448.9838    33   207392.0000    11820.0129 3.89e+08   94.30%
 25973355 25501284    21883.6636    33   207392.0000    11820.0279 3.89e+08   94.30%
Elapsed time = 188682.61 sec. (70111125.75 ticks, tree = 85733.89 MB, solutions = 1)
Nodefile size = 85605.64 MB (8364.42 MB after compression)
 25982488 25510197    23406.9047    37   207392.0000    11820.2149 3.89e+08   94.30%
 25992703 25520262   111140.2318    24   207392.0000    11820.4710 3.89e+08   94.30%
 26004478 25531838    24797.8353    29   207392.0000    11820.5560 3.89e+08   94.30%
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 26021413 25548479    25846.1198    35   207392.0000    11821.0349 3.90e+08   94.30%
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Extremely Challenging for MIP solvers

1. lightweight MIP model

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2. cutting planes

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2. cutting planes

3. symmetry handling

How did we solve it?

Lightweight MIP model

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} w_{ijk}$$

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$$w_{ijk} \geq \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} c_{ijkpqr} x_{pqr} - M(1 - x_{ijk})$$

$$x_{ijk} \in \{0, 1\}$$

$$w_{ijk} \geq 0$$

Lightweight MIP model

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superweak dual bound

Cutting Planes I

$$w_{ijk} \geq L_{ijk} x_{ijk}$$

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computed by solving a **LINEAR**
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- NP-hard in theory, quite cheap in practice
- can exploit additional constraints (if available) both global and local

Cutting Planes I

$$L_{ijk} = \begin{cases} \min \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} c_{ijkpqr} x_{pqr} \\ \sum_{p=1}^{n} \sum_{q=1}^{n} x_{pqr} = 1 \quad \forall r \in \{1, \ldots, n\} \\ \sum_{p=1}^{n} \sum_{r=1}^{n} x_{pqr} = 1 \quad \forall q \in \{1, \ldots, n\} \\ \sum_{q=1}^{n} \sum_{r=1}^{n} x_{pqr} = 1 \quad \forall p \in \{1, \ldots, n\} \\ x_{ijk} = 1 \\ x_{pqr} \in \{0, 1\} \end{cases}$$

Cutting Planes II

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computed by solving a MIP

- increase consistency between pairs of artificial variables
- not significantly harder than family I
- need to be conservative with separation

Cutting Planes II

$$
T_{ijkpqr} = \left\{
\begin{array}{l}
\min \quad w_{ijk} + w_{pqr} \\
\displaystyle\sum_{s=1}^{n}\sum_{t=1}^{n} x_{pqr} = 1 \quad \forall u \in \{1,\ldots,n\} \\
\displaystyle\sum_{s=1}^{n}\sum_{u=1}^{n} x_{pqr} = 1 \quad \forall t \in \{1,\ldots,n\} \\
\displaystyle\sum_{t=1}^{n}\sum_{u=1}^{n} x_{pqr} = 1 \quad \forall s \in \{1,\ldots,n\} \\
w_{ijk} \geq \displaystyle\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n} c_{ijkstu} x_{stu} \\
w_{pqr} \geq \displaystyle\sum_{s=1}^{n}\sum_{t=1}^{n}\sum_{u=1}^{n} c_{pqrstu} x_{stu} \\
x_{ijk} = 1 \\
x_{pqr} = 1 \\
x_{stu} \in \{0,1\}
\end{array}
\right.
$$

Symmetry Handling

binary variables can be partitioned into 6 orbits

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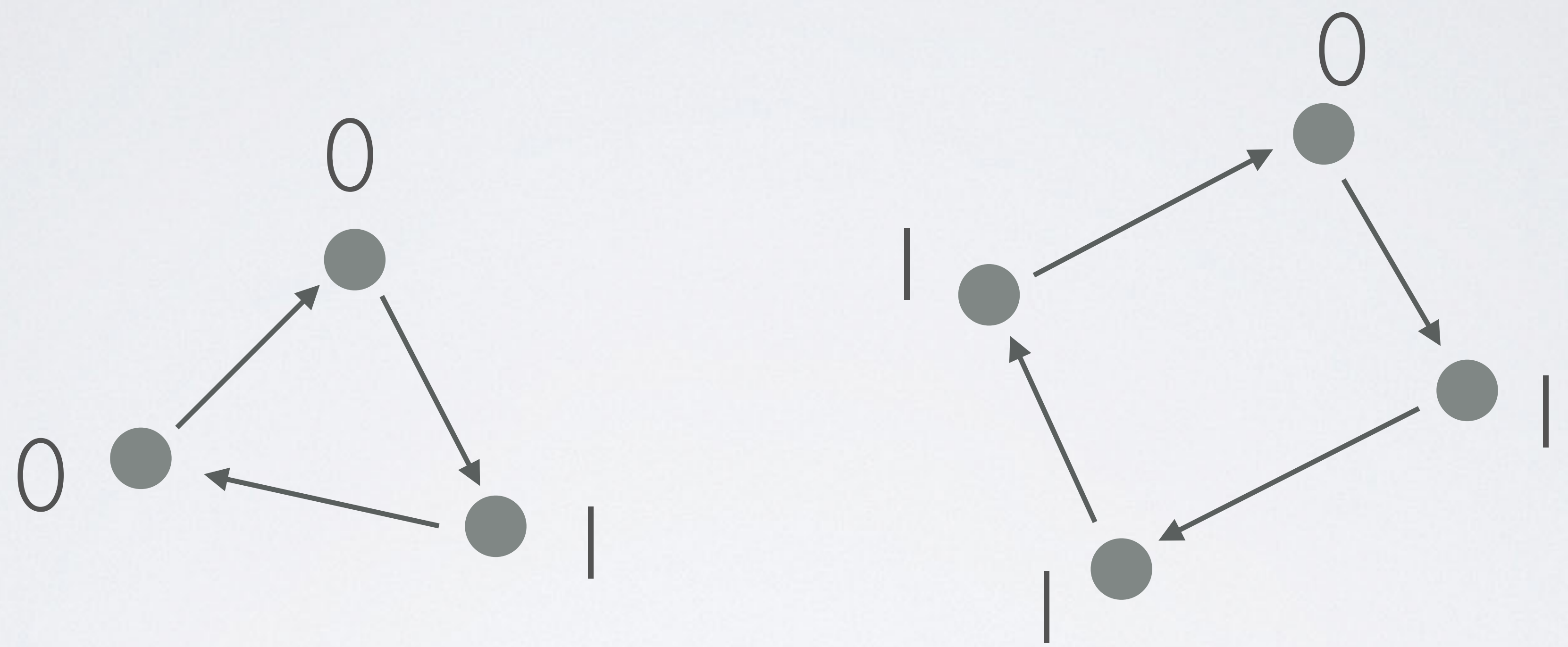

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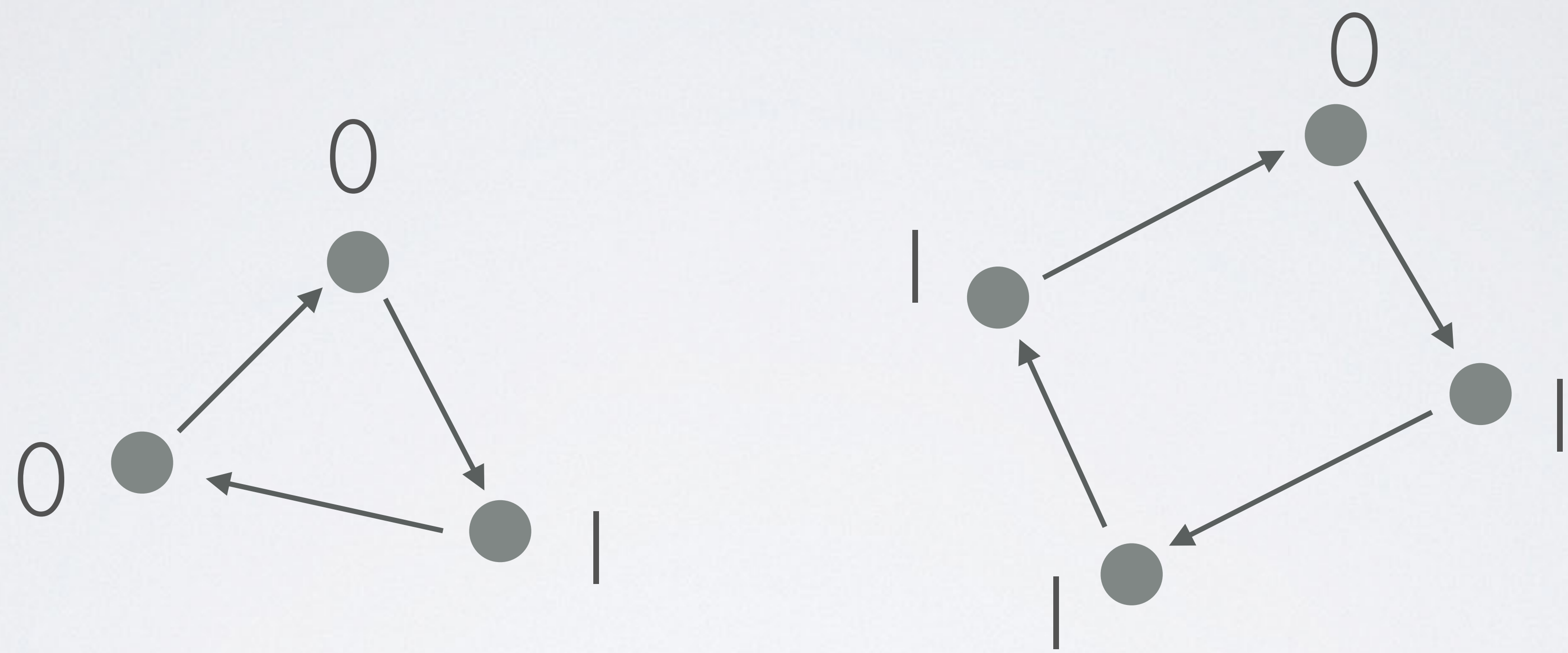

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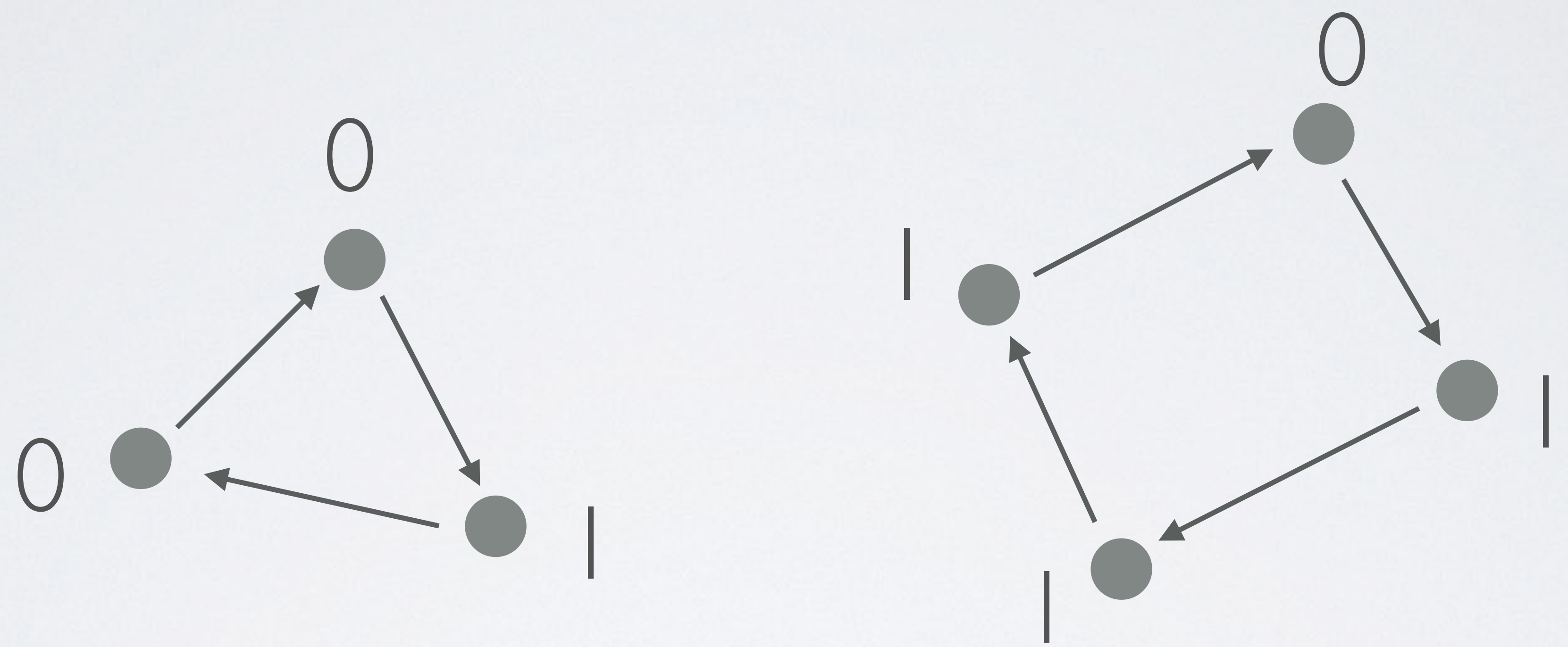

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aggregated variables as first level decisions

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isomorphism pruning within sub-MIPs…
exploit symmetry twice!!!

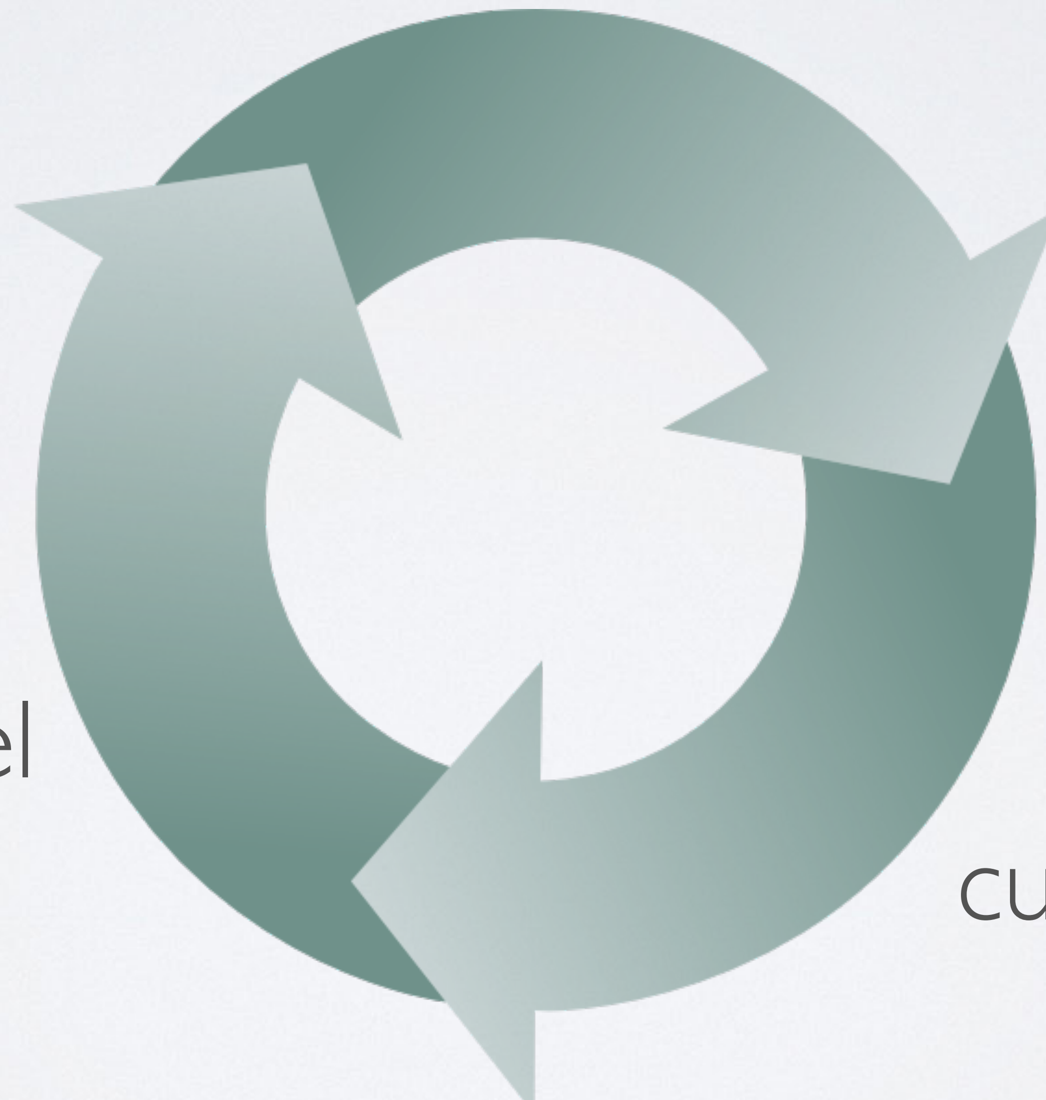
Aggregated Model

$$\begin{cases} y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 16 \\ y_1 + 2y_3 + y_4 = 16 \\ y_2 + y_4 + 2y_5 = 8 \\ 2y_0 + y_1 + y_2 = 8 \\ y_i \in \{0, \ldots, |O_i| - 1\} \quad \forall i \in \{0, \ldots, 5\} \end{cases}$$

6 orbits $\longmapsto$ 6 y variables

Nice Interplay between techniques

symmetry decomposition



MIP model

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Is it enough?

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NO!

Primal Heuristics

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$$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15$$
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- Plays well with isomorphism pruning

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- expensive cuts are separated only if one of the two controlling variables is already fixed to 1
- used indicator constraints to speed up LPs

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- primal solutions are evaluated with the exact coefficients

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less than one week on a desktop PC!

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- had fun :-)
- developed (extended) techniques that can be used for other Q3APs and (more importantly) QAPs and beyond…

Selected Literature

- Pierskalla: *The multi-dimensional assignment problem*. Operations Research 16, 422–431 (1968)
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- Margot: *Symmetry in Integer Linear Programming*. 50 Years of Integer Programming 1958-2008, 647–686 (2010)

Thanks for your attention!

Questions?