Recent developments in primal integer programming with applications on stochastic IP

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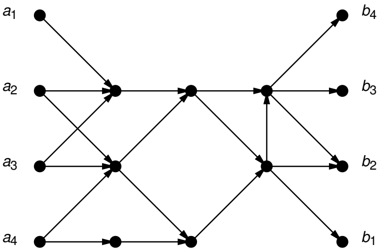
Based on the IPCO 2010 paper
A polynomial-time algorithm for N-fold 4-block decomposable integer programs
with Raymond Hemmecke (TU München)
and Robert Weismantel (ETH Zürich)

Sample application

2-stage stochastic integer multi-commodity flow

- *M* integer commodities
- known deterministic demand
- uncertain capacities per edge of the network
- known discrete probability distribution (*N* scenarios)
- 1[st] **stage:** decide how to transport the commodities over the given network without knowing the true capacities per edge
- 2[nd] **stage:** after observing the true capacities per edge, penalties have to be paid if capacity is exceeded
- **Objective:** minimize flow costs plus expected penalties



We study the computational complexity of the problem for an arbitrary **fixed network**, when only the parameters *N* and *M* are considered input.

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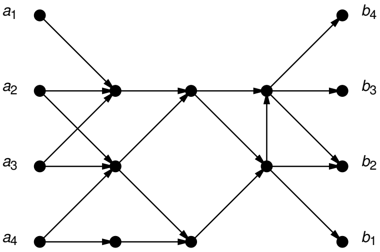

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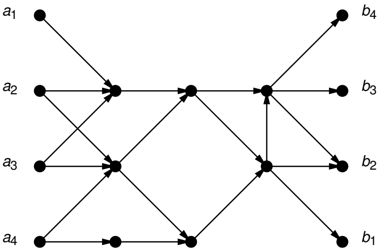


We study the computational complexity of the problem for an arbitrary **fixed network**, when only the parameters N and M are considered input.

Complexity result

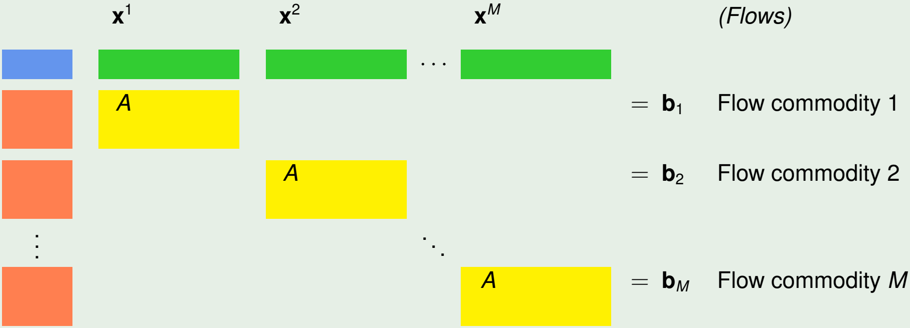
Theorem

For any given **fixed network** the 2-stage stochastic integer linear multi-commodity flow problem is **solvable in polynomial time**

- in the number M of commodities,
- in the number N of scenarios, and
- in the binary encoding lengths of the input data.

IP Formulation

Assume $M = N$.



All variables non-negative integer.

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$\mathbf{x}^{\text{agg}}$	$\mathbf{x}^1$	$\mathbf{x}^2$		$\mathbf{x}^M$		*(Flows)*
$-I$	I	I	$\cdots$	I	$= \mathbf{0}$	Flow aggregation
	A				$= \mathbf{b}_1$	Flow commodity 1
		A			$= \mathbf{b}_2$	Flow commodity 2
$\vdots$			$\ddots$			
				A	$= \mathbf{b}_M$	Flow commodity M

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I	$I \quad -I$				$= \mathbf{u}_1$	Capacity scenario 1
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$\vdots$			$\ddots$			
				A	$= \mathbf{b}_M$	Flow commodity M
I				$I \quad -I$	$= \mathbf{u}_N$	Capacity scenario N
	$\mathbf{s}_1 \quad \mathbf{t}_1$	$\mathbf{s}_2 \quad \mathbf{t}_2$		$\mathbf{s}_N \quad \mathbf{t}_N$		*(Slack/Excess)*

All variables non-negative integer.

N-fold 4-block decomposable IPs

In general, we study:

$$\min\left\{ f(\mathbf{z}) : \begin{pmatrix} C & D \\ B & A \end{pmatrix}^{(N)} \mathbf{z} = \mathbf{b},\ \mathbf{l} \leqslant \mathbf{z} \leqslant \mathbf{u},\ \mathbf{z} \in \mathbb{Z}^{n_B + N n_A} \right\}$$

where

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Applications:

- 2-stage stochastic integer multi-commodity flow (many variants)
- stochastic IPs with second-order dominance constraints (Gollmer–Gotzes–Schultz, 2009), equal probability case

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Two special cases

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Theorem (De Loera, Hemmecke, Onn, Rothblum, Weismantel, 2009)

Let $A \in \mathbb{Z}^{d_A \times n}$ and $D \in \mathbb{Z}^{d_D \times n}$ be **fixed**. Then the nonlinear IP

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Main new result (1)

Theorem: Poly-time separable convex augmentation

Let A, B, C, D be **fixed matrices** and f **separable convex** given by a comparison oracle.

$$(\mathrm{IP})_{N,\mathbf{b},\mathbf{l},\mathbf{u},f}: \qquad \min\left\{ f(\mathbf{z}) : \begin{pmatrix} C & D \\ B & A \end{pmatrix}^{(N)} \mathbf{z} = \mathbf{b},\ \ \mathbf{l} \leqslant \mathbf{z} \leqslant \mathbf{u},\ \ \mathbf{z} \in \mathbb{Z}^{n_B + N n_A} \right\}.$$

There exists an algorithm that, given a feasible solution $\mathbf{z}_0$ to $(\mathrm{IP})_{N,\mathbf{b},\mathbf{l},\mathbf{u},f}$,

- decides whether $\mathbf{z}_0$ is optimal or
- finds a better feasible solution $\mathbf{z}_1$ to $(\mathrm{IP})_{N,\mathbf{b},\mathbf{l},\mathbf{u},f}$ with $f(\mathbf{z}_1) < f(\mathbf{z}_0)$

that runs in time **polynomial** in N, in the binary encoding lengths $\langle \mathbf{l}, \mathbf{u}, \mathbf{b} \rangle$.

Main new result (2)

Corollary: Poly-time integer linear optimization

Let A, B, C, D be **fixed matrices** and f **linear**.

$$(\mathrm{IP})_{N,\mathbf{b},\mathbf{l},\mathbf{u},f}: \qquad \min\left\{ f(\mathbf{z}) : \begin{pmatrix} C & D \\ B & A \end{pmatrix}^{(N)} \mathbf{z} = \mathbf{b},\ \ \mathbf{l} \leqslant \mathbf{z} \leqslant \mathbf{u},\ \ \mathbf{z} \in \mathbb{Z}^{n_B + N n_A} \right\}.$$

There exists an algorithm that

- decides whether $(\mathrm{IP})_{N,\mathbf{b},\mathbf{l},\mathbf{u},f}$ is infeasible or unbounded, or
- finds an optimal solution to $(\mathrm{IP})_{N,\mathbf{b},\mathbf{l},\mathbf{u},f}$,

that runs in time **polynomial** in N and the binary encoding lengths $\langle \mathbf{l}, \mathbf{u}, \mathbf{b}, f \rangle$.

The machinery behind the proof:

Augmentation algorithms,
structured Graver bases

Augmentation vs. Optimization

For **linear integer optimization**:

1. Augmentation and Optimization are **poly-time equivalent** for 0/1 optimization (Schulz–Weismantel–Ziegler, 1990s)

$$(\mathrm{AUG}): \quad \mathbf{c}^\intercal \mathbf{t} < 0, \quad \mathbf{x} + \mathbf{t} \text{ feasible}$$

   - Modeled after scaling algorithms in combinatorial optimization

2. Directed Augmentation and Optimization are **poly-time equivalent** for general integer optimization (Schulz–Weismantel, 1990s)

$$(\mathrm{DIR\text{-}AUG}): \quad f(\mathbf{t}) = \mathbf{c}^\intercal \mathbf{t}^+ + \mathbf{d}^\intercal \mathbf{t}^- < 0, \quad \mathbf{x} + \mathbf{t} \text{ feasible}$$

   - Modeled after max mean cycle augmentation in min-cost flow
   - Note $f(\mathbf{t})$ is a **separable convex function** if $\mathbf{c} \geqslant \mathbf{0}$, $\mathbf{d} \geqslant \mathbf{0}$.

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Augmentation with Graver bases

The **Graver basis** $\mathcal{G}(A)$ of a matrix $A \in \mathbb{Z}^{d\times n}$ is the set of all non-zero vectors

$$\mathbf{z} \in \ker(A) \cap \mathbb{Z}^n = \{\mathbf{v} : A\mathbf{v} = \mathbf{0}, \mathbf{v} \in \mathbb{Z}^n\}$$

that **cannot** be written ("irreducible") as

$$\mathbf{z} = \mathbf{x} + \mathbf{y}$$

with non-zero vectors $\mathbf{x}, \mathbf{y} \in \ker(A) \cap \mathbb{Z}^n$ with the same sign-pattern from $\{\leqslant \mathbf{0}, \geqslant \mathbf{0}\}^n$ as $\mathbf{z}$.

Facts

- The **Graver basis** $\mathcal{G}(A)$ of a matrix $A \in \mathbb{Z}^{d\times n}$ is always finite.
- Every $\mathbf{z} \in \ker(A) \cap \mathbb{Z}^n$ has a sign-compatible representation

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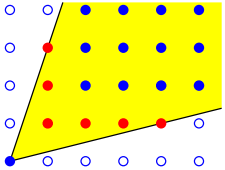
Graver bases and Hilbert bases

Consider the cones $C = \ker A \cap \mathcal{O}_j$ for all orthants $\mathcal{O}_j$ of $\mathbb{R}^n$.



Hilbert basis

The **Hilbert basis** $\mathcal{H}(C) \subseteq C \cap \mathbb{Z}^n$ of a pointed cone C is the unique inclusion-minimal set such that each $\mathbf{z} \in C \cap \mathbb{Z}^n$ can be written as

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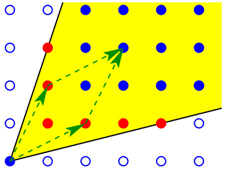

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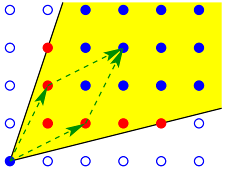

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Theorem (Graver, Murota, Saito, Weismantel, Hemmecke, De Loera, Onn, Rothblum)

Graver bases provide **optimality certificates**, i.e.,

- solve (AUG) for **linear integer** optimization problems,
- solve (DIR-AUG) for **linear integer** optimization problems,
- solve (AUG) for **separable convex integer minimization problems**

$$\min\left\{\sum_{i=1}^{n} f_i(z_i) : A\mathbf{z} = \mathbf{b},\ \mathbf{l} \leqslant \mathbf{z} \leqslant \mathbf{u},\ \mathbf{z} \in \mathbb{Z}^n\right\},$$

and also provide edge directions for the integer hull and thus:

- solve (OPT) for **convex integer maximization problems**

$$\min\{f(W\mathbf{z}) : A\mathbf{z} = \mathbf{b},\ \mathbf{l} \leqslant \mathbf{z} \leqslant \mathbf{u},\ \mathbf{z} \in \mathbb{Z}^n\}$$

(where W is a linear map into fixed dimension)

In particular, if a Graver basis is known (and efficient), using (DIR-AUG) we can solve the IP in **polynomial time**.

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- solve (AUG) for **separable convex integer minimization problems**

$$\min \left\{ \sum_{i=1}^{n} f_i(z_i) : A\mathbf{z} = \mathbf{b},\ \mathbf{l} \leqslant \mathbf{z} \leqslant \mathbf{u},\ \mathbf{z} \in \mathbb{Z}^n \right\},$$

and also provide edge directions for the integer hull and thus:

- solve (OPT) for **convex integer maximization problems**

$$\min\{f(W\mathbf{z}) : A\mathbf{z} = \mathbf{b},\ \mathbf{l} \leqslant \mathbf{z} \leqslant \mathbf{u},\ \mathbf{z} \in \mathbb{Z}^n\}$$

(where W is a linear map into fixed dimension)

In particular, if a Graver basis is known (and efficient), using (DIR-AUG) we can solve the IP in **polynomial time**.

Greedy Graver Augmentation

Integer Carathéodory Property (Sebö)

$2n-2$ sign-compatible Graver elements (with integer multiplicity) suffice to give a feasible path from any feasible solution to any optimal solution

Greedy Graver Augmentation:

(GG-AUG): Choose Graver element $\mathbf{t}$, $\lambda \in \mathbb{Z}_+$ that maximize $f(\mathbf{x} + \lambda \mathbf{t})$ s.t. $\mathbf{x} + \lambda \mathbf{t}$ feasible.

Geometric Improvement: Each step closes at least $\frac{1}{2n-2}$ of the remaining optimality gap. So after $\log(\text{initial gap})$ steps, gap is < 1, so optimal.

Theorem (De Loera, Hemmecke, Onn, Rothblum, Weismantel, 2009)

Optimization (for linear and separable convex minimization) is poly-time solvable relative to Greedy Graver Augmentation oracles.

Unfortunately, we don't know how to use it for our problem. **(Open problem)**

So we're back to DIR-AUG, which restricts us to the linear case.

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How do we get poly-time algorithms?

Concave and separable convex N-fold IPs

Problem matrix

$$\begin{pmatrix} \mathbf{0} & D \\ \mathbf{0} & A \end{pmatrix}^{(N)} = \begin{pmatrix} D & D & \cdots & D \\ A & 0 & & 0 \\ 0 & A & & 0 \\ & & \ddots & \\ 0 & 0 & & A \end{pmatrix}$$

Theorem (Santos, Sturmfels; Hoşten, Sullivant)

If A and D are fixed matrices, then

$$\max \left\{ \|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}\left(\begin{pmatrix} \mathbf{0} & D \\ \mathbf{0} & A \end{pmatrix}^{(N)} \right) \right\} \in O_N(1).$$

Corollary (De Loera, Hemmecke, Onn, Weismantel, 2009)

Let $A \in \mathbb{Z}^{d_A \times n}$ and $D \in \mathbb{Z}^{d_D \times n}$ be **fixed**. Then the sizes of the Graver bases increase only **polynomially** in N.

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is solvable in **polynomial time** for **separable convex** f and for **concave** $f = g \circ W$.

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Separable convex 2-stage stochastic IPs

Problem matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & A \end{pmatrix}^{(N)} := \begin{pmatrix} B & A & 0 & & 0 \\ B & 0 & A & & 0 \\ \vdots & & & \ddots & \\ B & 0 & 0 & & A \end{pmatrix}$$

Theorem (Hemmecke, Schultz, 2002)

If A and B are **fixed** matrices, then

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Result extends to the multi-stage situation.

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How can we combine both results?

General idea

Theorem (Hemmecke, **Kö.**, Weismantel, 2010)

If A, B, C, D are **fixed** matrices, then

$$\max\left\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}\left(\begin{pmatrix} C & D \\ B & A \end{pmatrix}^{(N)}\right)\right\}$$

is bounded by a **polynomial** in N.

Problem matrix

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Reconstruction process

Let $\mathbf{z}_0$ be a non-optimal feasible solution and let $\mathbf{t} = (\mathbf{u}, \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(N)})$ denote the building blocks of an augmenting vector (to be found).

- Only **polynomially many** choices for $\mathbf{u}$.
- For each fixed $\mathbf{u}$, we find the best choice for $(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(N)})$ by solving an N-fold IP with problem matrix $\begin{pmatrix} \mathbf{0} & D \\ \mathbf{0} & A \end{pmatrix}^{(N)}$.

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Polynomial bound on the 1-norm (1)

Primitive partition identities (Diaconis, Graham, Sturmfels, 1996)

The maximum 1-norm of a Graver basis element of $(1\ 2\ \ldots\ n)$ is $2n-1$.

Lemma

Let $A \in \mathbb{Z}^{d\times n}$, $\mathbf{a} \in \mathbb{Z}^n$ and put $C := \binom{\mathbf{a}^\intercal}{A}$. Moreover, let $M = \max\{|a_i|\}$. Then we have

$$\max\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}(C)\} \leqslant 2nM \max\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}(A)\}^2.$$

Corollary

Let $A \in \mathbb{Z}^{d_A\times n}$, $B \in \mathbb{Z}^{d_B\times n}$ and put $C := \binom{B}{A}$. Moreover, let $M = \max\{|B_{ij}|\}$. Then we have

$$\max\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}(C)\} \leqslant (2nM)^{2^{d_B}-1} \left(\max\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}(A)\}\right)^{2^{d_B}}.$$

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Bounding the 1-norm when adding a single row

Proof of the Lemma

- Let $\mathbf{z} \in \mathcal{G}(C)$. Then $\mathbf{z} \in \ker(A) \cap \mathbb{Z}^n$ and thus it can be written as
$$\mathbf{z} = \sum \lambda_i \mathbf{g}_i =: \mathcal{G}(A)\lambda, \quad \lambda \in \mathbb{Z}_+^{|\mathcal{G}(A)|}$$
- Moreover, $C\mathbf{z} = \mathbf{0}$ is equivalent to $\mathbf{a}^\intercal(\mathcal{G}(A)\lambda) = (\mathbf{a}^\intercal\mathcal{G}(A))\lambda = 0$.
- If there is some $\mu \in \mathbb{Z}_+^{|\mathcal{G}(A)|}$ with $(\mathbf{a}^\intercal\mathcal{G}(A))\mu = 0$, $\mathbf{0} \neq \mu \neq \lambda$ and $\mu \leqslant \lambda$, then
$$\mathbf{z} = \mathcal{G}(A)\mu + \mathcal{G}(A)(\lambda - \mu)$$
contradicts $\mathbf{z} \in \mathcal{G}(C)$.
- Thus, $\mathbf{z} \in \mathcal{G}(C)$ implies that λ is an indecomposable solution to
$$[\mathbf{a}^\intercal\mathcal{G}(A)]\lambda = 0, \quad \lambda \in \mathbb{Z}_+^{|\mathcal{G}(A)|}.$$
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Bounding the 1-norm when adding a single row

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- Let $\mathbf{z} \in \mathcal{G}(C)$.Then $\mathbf{z} \in \ker(A) \cap \mathbb{Z}^n$ and thus it can be written as
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Polynomial bound on the 1-norm (2)

Theorem (Hemmecke, Schultz, 2002)

If A, B are fixed matrices, then

$$\max\left\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}\left(\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & A \end{pmatrix}^{(N)}\right)\right\} \in O_N(N).$$

Corollary (Hemmecke, **Kö.**, Weismantel, 2010)

If A, B, C, D are fixed matrices, then

$$\max\left\{\|\mathbf{v}\|_1 : \mathbf{v} \in \mathcal{G}\left(\begin{pmatrix} C & D \\ B & A \end{pmatrix}^{(N)}\right)\right\} \in O_N\left(N^{2^{d_B}}\right),$$

i.e., bounded **polynomially** in N.

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Some open questions, and end of the talk

- Open: Greedy Graver for 4-block problems
- Hemmecke, Onn, Romanchuk (2011) give the first algorithm for N-fold linear integer programming, which runs in
  - **cubic time** for fixed A, D,
  - pseudo-polynomial time when the matrices A, D are part of the input, but of fixed dimension,

  using Graver bases and parametric dynamic programming.

  Open:
  - General solutions for convex integer minimization
  - Generalization to 4-block problems
  - Faster algorithms
- Open: More general results for structured IP