Deterministic Compressed Sensing for Images using Chirps and Reed-Muller Sequences

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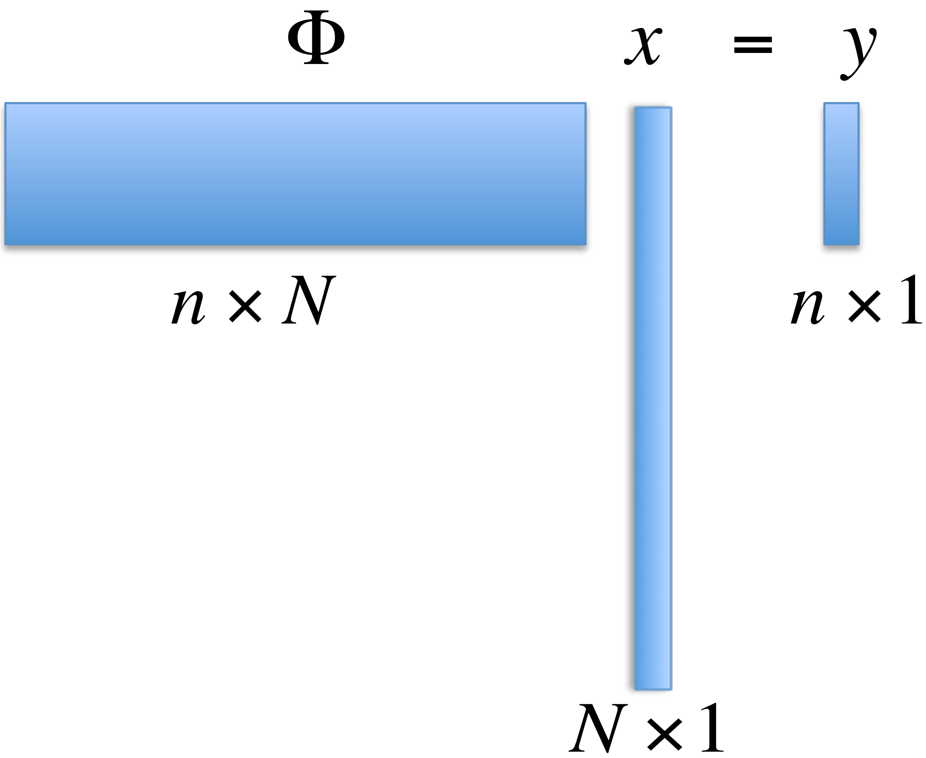
Outline

- Introduction
  - Deterministic CS using chirps and Reed-Muller sequences
- Motivation
  - Need a method suitable for images
- Method
  - Construction of the CS matrix
  - Reconstruction algorithm
    - Best initial approximation
    - Fast chirp transforms
- Results

Compressed Sensing*: overview



Φ	sensing matrix
y	data/measurements
x	sparse signal

want to recover

1. x : k-sparse $\qquad k < n << N$
2. Φ : RIP (Restricted Isometry Property), e.g. random matrices
3. Practical reconstruction algorithms, e.g. ℓ_1 minimization

*Candes, Romberg, Tao 2006 and Donoho 2006

Motivation

- Random matrices (Gaussian, …) satisfy RIP
  high probability of successful recovery

- Why deterministic sensing
  - Explicit reconstruction algorithm
  - Efficient storage
  - Smaller error in reconstruction

- Existing works
  - Chirp matrices – Applebaum, Howard, Searle, Calderbank
  - 2nd-order RM sequences – Howard, Calderbank, Searle, Jafarpour
  - DeVore, Indyk, Iwen, Herman, …

- Need a method suitable for images

Statistical Restricted Isometry Property*

Φ is $(k,\ \varepsilon,\ \delta)$-StRIP if for *k*-sparse $x \in R^N$

$$(1-\varepsilon)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1+\varepsilon)\|x\|_2^2$$

holds with prob. exceeding 1-δ

(w.r.t. a uniform distribution of x among all *k*-sparse vectors)

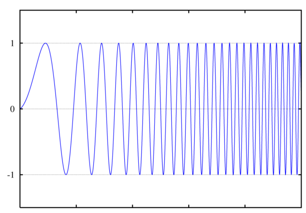
*Calderbank, Howard, Jafarpour,
Construction of a Large Class of Deterministic Sensing Matrices that Satisfy a Statistical Isometry Property

CS with Chirps and RM Sequences

- discrete chirp signal

$$\upsilon_{r,m}(l) = \frac{1}{\sqrt{n}} e^{\frac{2\pi i}{n} ml + \frac{2\pi i}{n} rl^2}$$

$$r,\ m,\ l \in \mathrm{Z}_n$$



- complex sinusoids

$$\upsilon_{0,m}(l) = \frac{1}{\sqrt{n}} e^{\frac{2\pi i}{n} ml}$$

- 2nd-order Reed-Muller functions

$$\phi_{P,b}(a) = \frac{1}{\sqrt{2^m}} i^{2b^T a + (Pa)^T a}$$

$$a,\ b \in \mathrm{Z}_2^m \quad \text{(binary vectors of length } m\text{)}$$

$$P:\ m \times m \text{ binary symmetric matrix}$$

- Walsh functions

$$\phi_{0,b}(a) = \frac{1}{\sqrt{2^m}} i^{2b^T a}$$

CS with Chirps and RM Sequences

- inverse Fourier matrix

$n=3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}1\cdot 1} & e^{\frac{2\pi i}{3}2\cdot 1} \\ 1 & e^{\frac{2\pi i}{3}1\cdot 2} & e^{\frac{2\pi i}{3}2\cdot 2} \end{bmatrix}$$

- Fast Fourier Transform

- *chirp matrix

$$\Phi = \underbrace{[U_{r_1} \quad U_{r_2} \quad \cdots \quad U_{r_n}]}_{n \times n^2}$$

* *Applebaum et al.*

- Hadamard matrix

$m=2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

- Fast Hadamard Transform (FHT)

- **Reed-Muller matrix (P is zero-diagonal)

$$\Phi = \underbrace{[U_{P_1} \quad U_{P_2} \quad \cdots \quad U_{P_{2^{m(m-1)/2}}}]}_{2^m \times 2^{m(m+1)/2}}$$

** *Howard et al.*

Chirp Sensing Matrix

$$\Phi = \underbrace{[U_{r_1} \quad U_{r_2} \quad \cdots \quad U_{r_n}]}_{n \times n^2}$$

- n = 3

  size: 3 X 3^2

$$m = 0,\ 1,\ 2$$
$$l = 0,\ 1,\ 2$$

$$\Phi = [U_{r_1} \quad U_{r_2} \quad U_{r_3}] =$$

$$m \longrightarrow \qquad\qquad m \longrightarrow \qquad\qquad m \longrightarrow$$

$$l \downarrow \qquad \frac{1}{\sqrt{3}}\left[\underbrace{\begin{matrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}1\cdot 1} & e^{\frac{2\pi i}{3}2\cdot 1} \\ 1 & e^{\frac{2\pi i}{3}1\cdot 2} & e^{\frac{2\pi i}{3}2\cdot 2} \end{matrix}}_{r_1 = 0} \quad \underbrace{\begin{matrix} 1 & 1 & 1 \\ e^{\frac{2\pi i}{3}1\cdot 1^2} & e^{\frac{2\pi i}{3}1\cdot 1+\frac{2\pi i}{3}1\cdot 1^2} & e^{\frac{2\pi i}{3}2\cdot 1+\frac{2\pi i}{3}1\cdot 1^2} \\ e^{\frac{2\pi i}{3}1\cdot 2^2} & e^{\frac{2\pi i}{3}1\cdot 2+\frac{2\pi i}{3}1\cdot 2^2} & e^{\frac{2\pi i}{3}2\cdot 2+\frac{2\pi i}{3}1\cdot 2^2} \end{matrix}}_{r_2 = 1} \quad \underbrace{\begin{matrix} 1 & 1 & 1 \\ e^{\frac{2\pi i}{3}2\cdot 1^2} & e^{\frac{2\pi i}{3}1\cdot 1+\frac{2\pi i}{3}2\cdot 1^2} & e^{\frac{2\pi i}{3}2\cdot 1+\frac{2\pi i}{3}2\cdot 1^2} \\ e^{\frac{2\pi i}{3}2\cdot 2^2} & e^{\frac{2\pi i}{3}1\cdot 2+\frac{2\pi i}{3}2\cdot 2^2} & e^{\frac{2\pi i}{3}2\cdot 2+\frac{2\pi i}{3}2\cdot 2^2} \end{matrix}}_{r_3 = 2}\right]$$

RM Sensing Matrix

$$\Phi = \underbrace{[U_{P_1} \quad U_{P_2} \quad \cdots \quad U_{P_{2^{m(m-1)/2}}}]}_{2^m \times 2^{m(m+1)/2}}$$

- m = 2
  size: $2^2 \times 2^3$

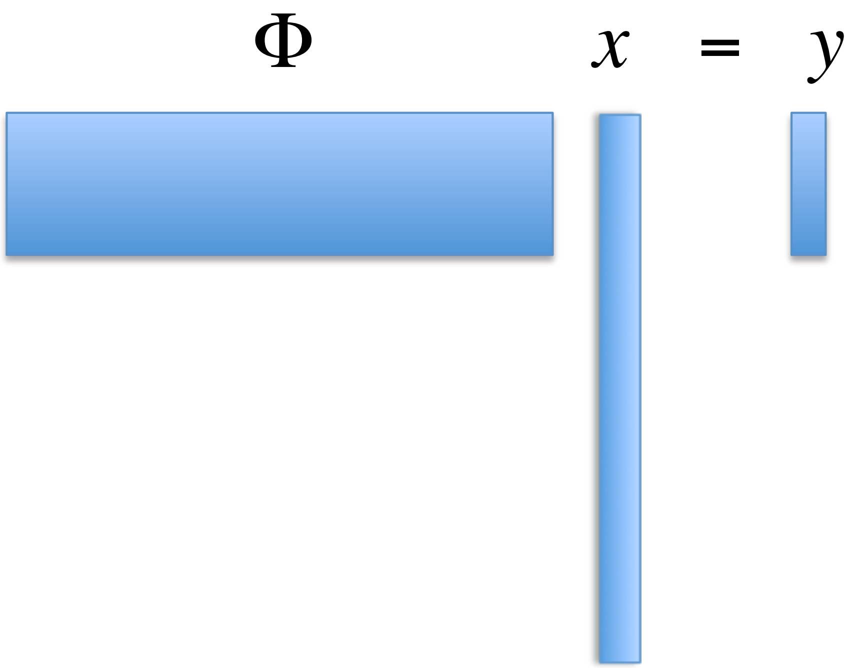
$$a, b \in Z_2^2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \qquad P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Phi = [U_{P_1} \quad U_{P_2}] = \frac{1}{2} \begin{array}{c} \\ \\ a\downarrow \end{array} \begin{bmatrix} \overset{b \rightarrow}{} & & & & \overset{b \rightarrow}{} & & & \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ \underbrace{1 \quad -1 \quad -1 \quad 1}_{P_1} & & & & \underbrace{1 \quad -1 \quad -1 \quad 1}_{P_2} & & & \end{bmatrix}$$

- m = 10 → $2^{10} \times 2^{55}$

Quadratic Reconstruction Algorithm

Chirp version*



$$y(l) = z_1 v_{r_1,m_1}(l) + \cdots + z_k v_{r_k,m_k}(l)$$

$$O(kn^2 \log n)$$

1. Find r_i -- shift-and-multiply & FFT, find peak $\quad \mathrm{FFT}\left\{\overline{y(l)}\, y(l+T)\right\}$

2. Find m_i -- dechirp & FFT, find peak $\quad \mathrm{FFT}\left\{y(l)\overline{v_{r_j,0}(l)}\right\}$

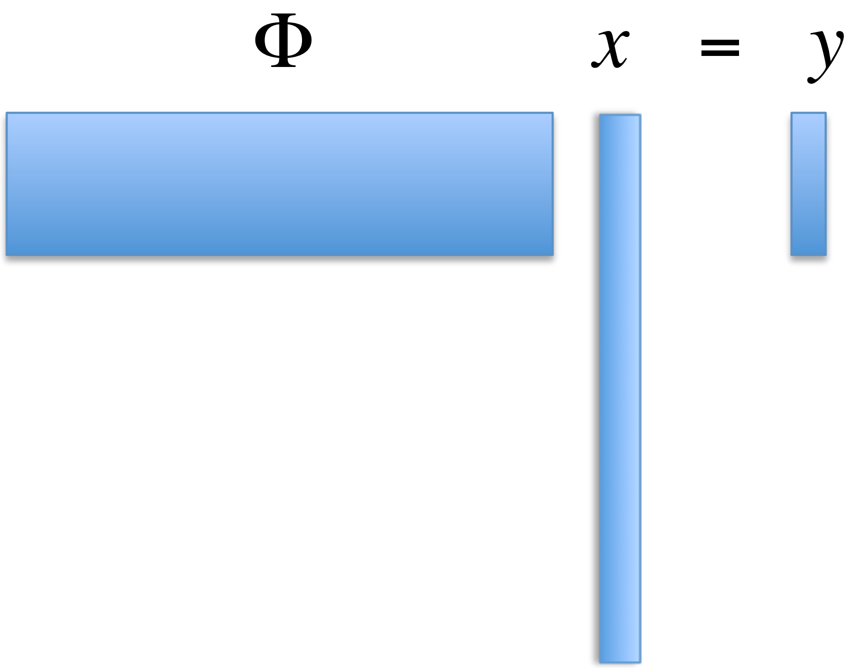
3. Find z_i -- least squares $\quad \min_z \left\| y(l) - \sum z_j\, v_{r_j,m_j}(l) \right\|^2$

4. Repeat 1. – 3. -- until residual < epsilon

* Applebaum *et al.*

Quadratic Reconstruction Algorithm

RM version*



$$y(a) = z_1\phi_{P_1,b_1}(a) + \cdots + z_k\phi_{P_k,b_k}(a)$$

Fast Hadamard Transform (FHT)
(projection onto Walsh basis)

$$O\left(kn(\log n)^2\right)$$

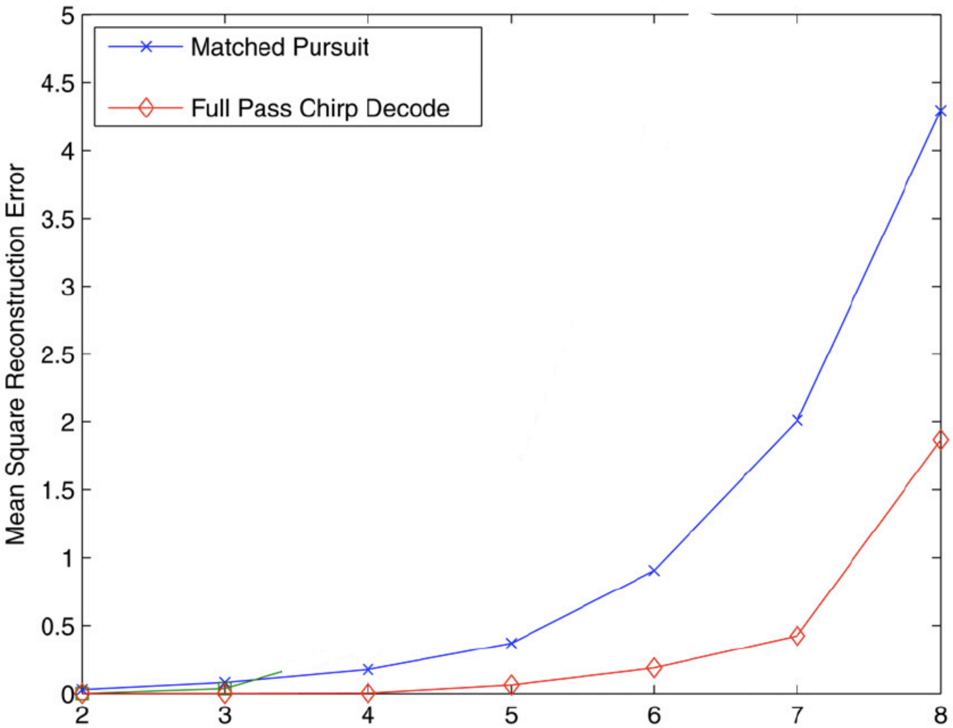
1. Find P_i -- shift-and-multiply & FHT, find peak $\mathrm{FHT}\left\{\overline{y(a)}\, y(a+e_j)\right\}$
2. Find b_i -- "dechirp" & FHT, find peak $\mathrm{FHT}\left\{y(a)\overline{\phi_{P_j,0}(a)}\right\}$
3. Find z_i -- least squares $\min_z \left\| y(a) - \sum z_j\, \phi_{P_j,b_j}(a) \right\|^2$
4. Repeat 1. – 3. -- until residual < epsilon

* Howard *et al.*

Chirp Results
comparison with MP + random matrices

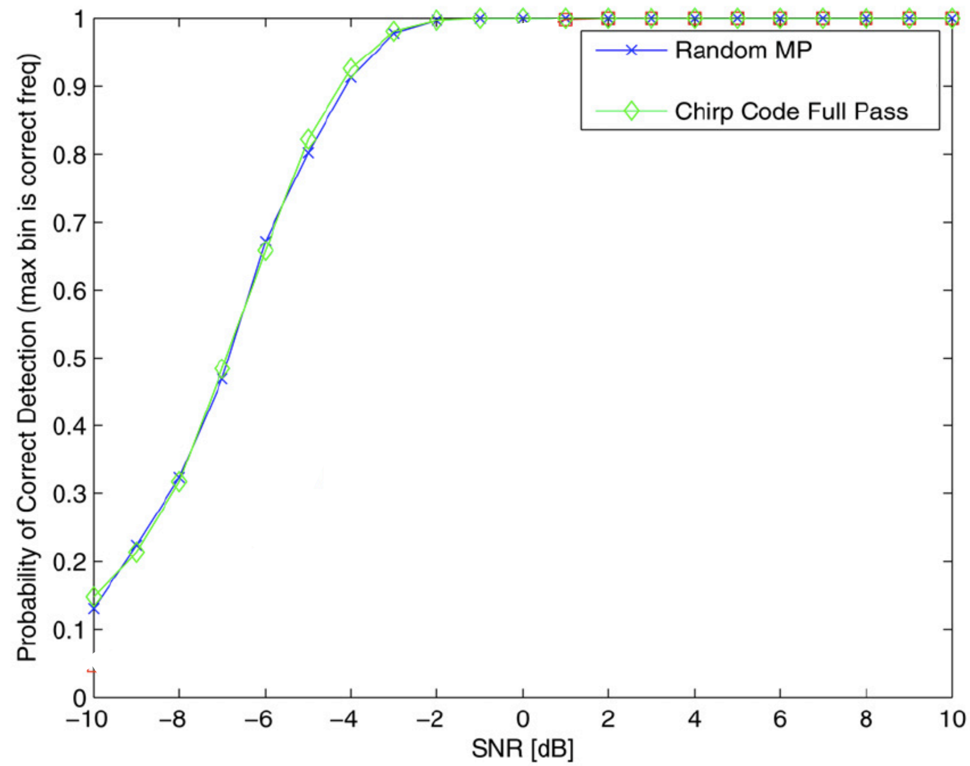
n=67

Reconstruction of sparse signals



Sparsity *k*

Reconstruction of sparse signals with noise

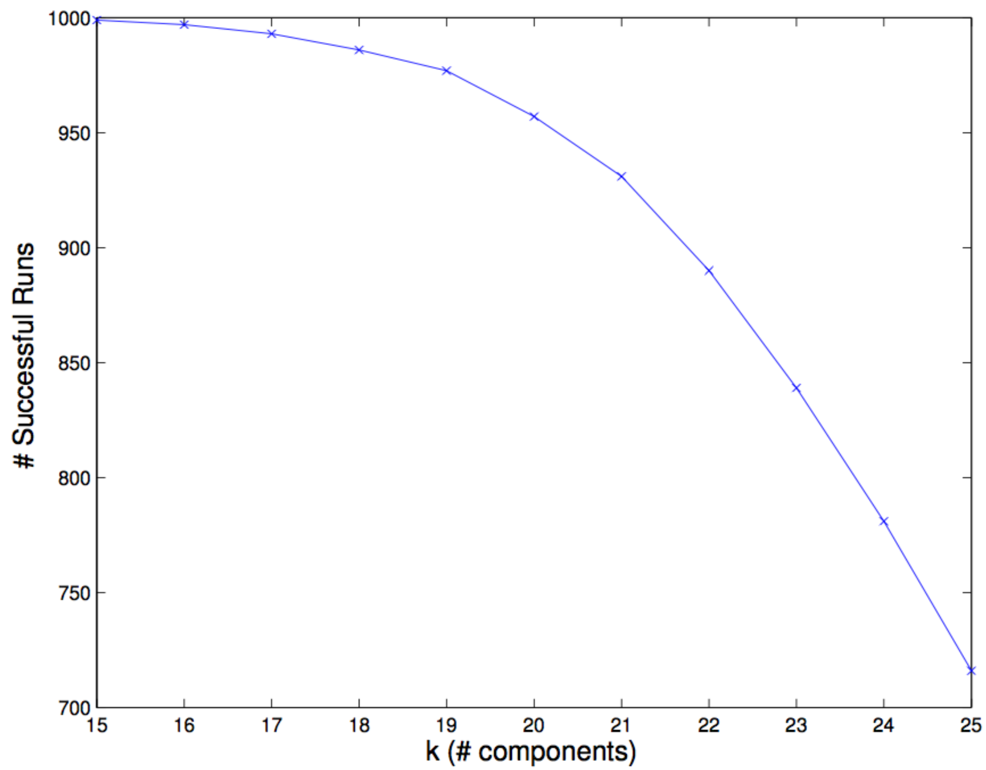


Matching Pursuit - - $O(knN)$

Deterministic CS with Chirp alg.- - $O(kn^2 \log n)$

* Applebaum *et al.*

RM Results



**Howard et al.

- Rule of thumb*: $n > k \log_2(1 + N/k)$
  (high prob. of successful recon. using ℓ_1 min. + random matrix)

- $m = 10 \Rightarrow 2^{10} \times 2^{55} \Rightarrow n = 1{,}024$ and $N = 3.6 \times 10^{16}$
  by rule of thumb $\Rightarrow k < 20$

*Baron et al., Donoho and Tanner

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Chirp and RM Quadratic Reconstruction Algorithms pros & cons

k = sparsity,

n = #measurements

N = signal size

- Pros: Outperform MP in recon. error and computational complexity
  - MP $O(knN)$
  - det CS with RM $O\left(kn(\log n)^2\right)$

- Cons: Not suited for images

  256×256 image with 10% sparsity $\longrightarrow$ $k = 6{,}554,\ N = 65{,}536$

  - by rule of thumb, $n > k\log_2(1 + N/k)$ $\longrightarrow$ $n \approx 22{,}670$

    $$\frac{N}{n} = \frac{65536}{22670} \approx 2.89$$

    but $n \times N = 2^m \times 2^{m\,(m+1)/2}$ $\longrightarrow$ image not sparse enough

  - least squares problem becomes too large

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Construction of Sensing Matrices

chirp

- N = image size (expl: *256 X 256 = 2*^{16})
- n = prime greater than and closest to $N/4$ (expl: *16,411*)
- Sensing matrix: $\Phi = [U_{r_1} \quad U_{r_2} \quad U_{r_3} \quad \tilde{U}_{r_4}]$

$$\boxed{\begin{array}{c} \text{coherence} = \dfrac{1}{\sqrt{n}} \\ \left(= \max\limits_{j \neq l} \left| \left\langle \varphi_j, \varphi_l \right\rangle \right| \right) \end{array}}$$

RM

- N = image size (expl: *256 X 256 = 2*^{16})
- $n = N/4$ (expl: 2^{14} = 16,384)
- Sensing matrix: $\Phi = [U_{P_1} \quad U_{P_2} \quad U_{P_3} \quad U_{P_4}]$

Reconstruction Algorithm for Images

- Get initial best approximation solution

Repeat 1 - 3 until residual is sufficiently small

1 Find multiple (P_i, b_i) pairs

2 Determine z_i by least squares solutions

3 Get residual $y(a) = y(a) - \sum_{l=1}^{j} z_l \phi_{P_l, b_l}(a)$

Initial Best Approximation of Solution

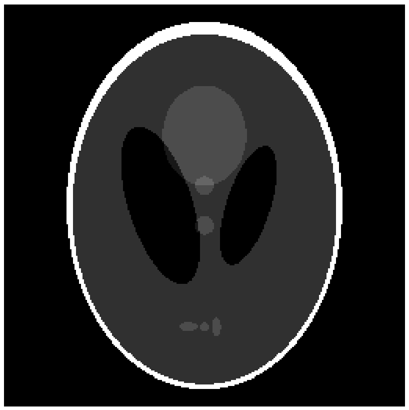
- Detection of the "bulk" of a signal
- Based on energy of wavelets concentrate on upper-left region

$$y = \Phi x = [U_{P_1} \quad U_{P_2} \quad U_{P_3} \quad U_{P_4}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = U_{P_1} x_1 + U_{P_2} x_2 + U_{P_3} x_3 + U_{P_4} x_4$$

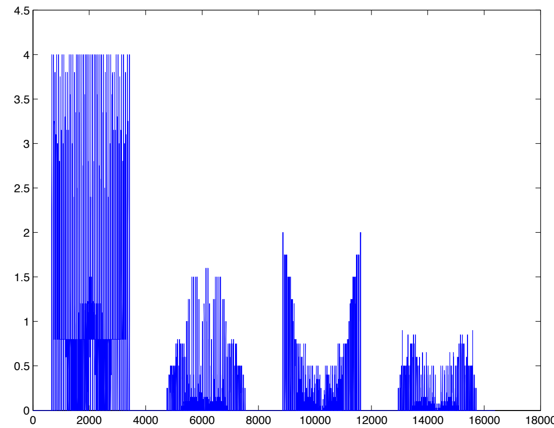
$$U_{P_1}^* y = U_{P_1}^* U_{P_1} x_1 + U_{P_1}^* U_{P_2} x_2 + U_{P_1}^* U_{P_3} x_3 + U_{P_1}^* U_{P_4} x_4$$

$$\approx x_1$$

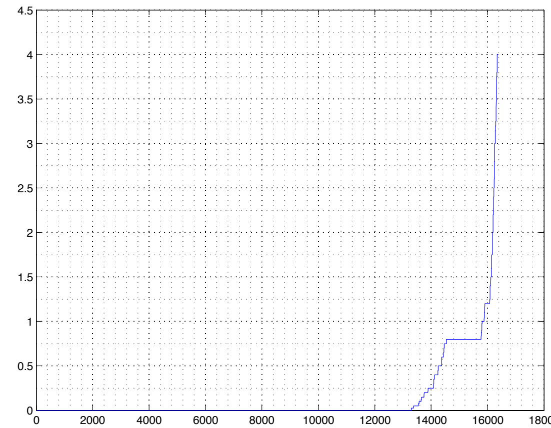
Example: Initial Best Approximation



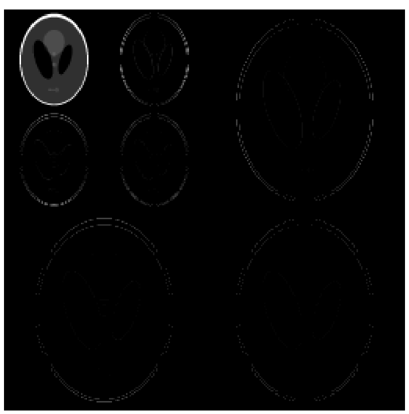
256 X 256 Shepp-Logan phantom image



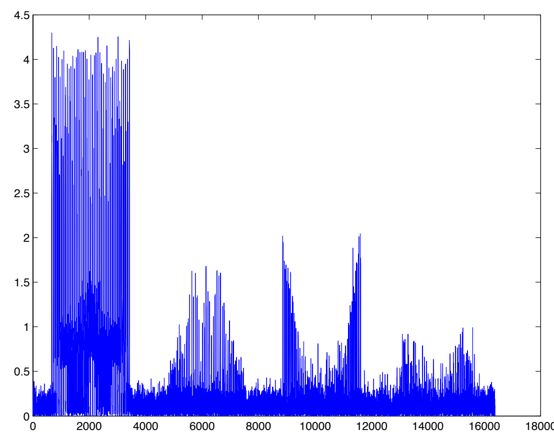
$|x_1|$



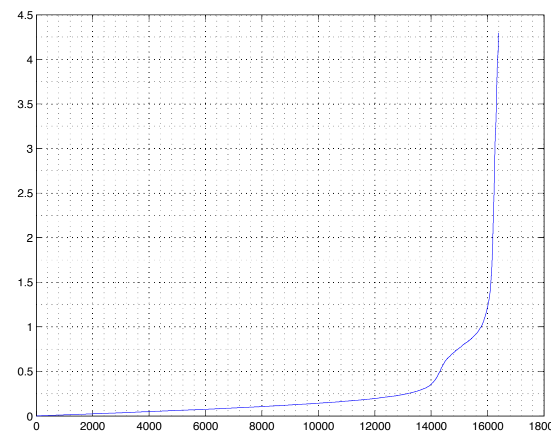
$|x_1|$ in ascending order



wavelet coefficients



$|U_{P_1}^T y|$



$|U_{P_1}^T y|$ in ascending order

- No prior knowledge of individual image required

Find Multiple $(P_i\,,\, b_i)$ Pairs

Discrete Chirp-Hadamard Transform (DCHT) : find $(P_i\,,\, b_i)$ pairs

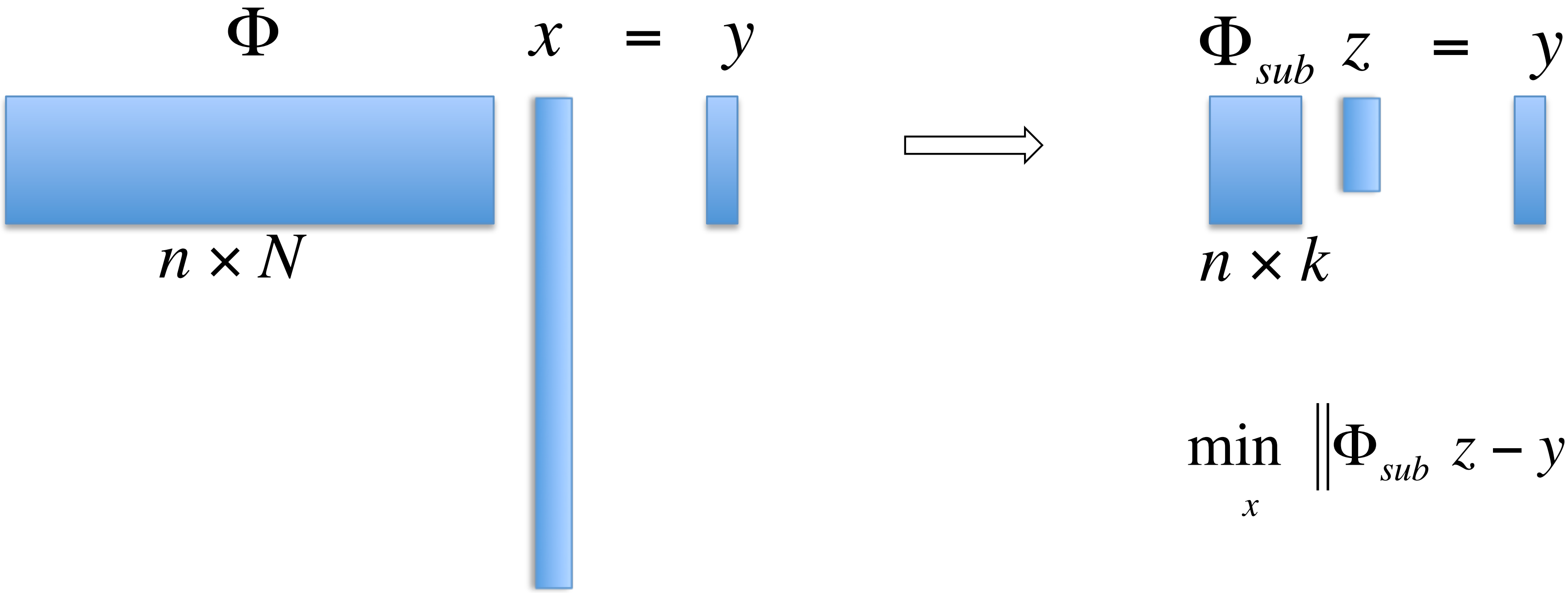
- DCHT = "de-chrip" + FHT
- Only 4 P-matrices used

$$\boxed{\mathrm{FHT}\left((-i)^{(P_j a)^T a}\, y(a)\right), \qquad j = 1,\ 2,\ 3,\ 4}$$

- Pick $d \approx 100$ peaks in the FHT
- Computational complexity improves from $O(kn(\log n)^2)$

to $O(\frac{1}{d} 4kn\log n)$

Determine z_i by Least Squares

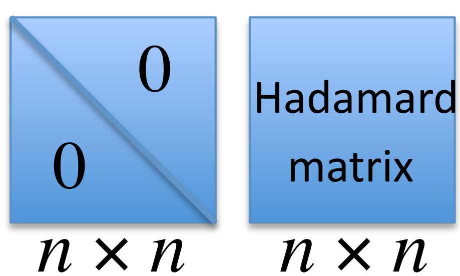


$$\min_{x} \left\|\Phi_{sub}\, z - y\right\|^2$$

solved by LSQR
[Paige & Saunders]

$$\Phi = [U_{P_1} \quad U_{P_2} \quad U_{P_3} \quad U_{P_4}]$$

Each U_{P_j} is



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Reconstruction SNR

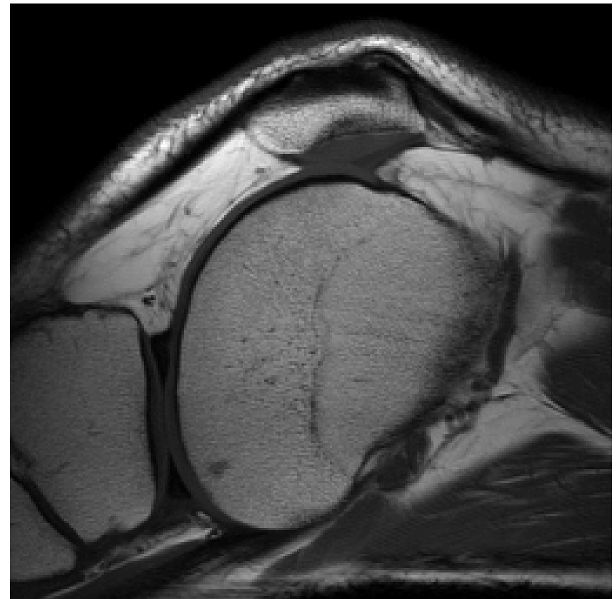
$$\text{SNR(dB)} = 10 \log_{10}\left[\frac{\| x_{\text{actual}} \|^2}{\| x_{\text{actual}} - x_{\text{reconstructed}} \|^2}\right]$$

256 X 256 images

cameraman

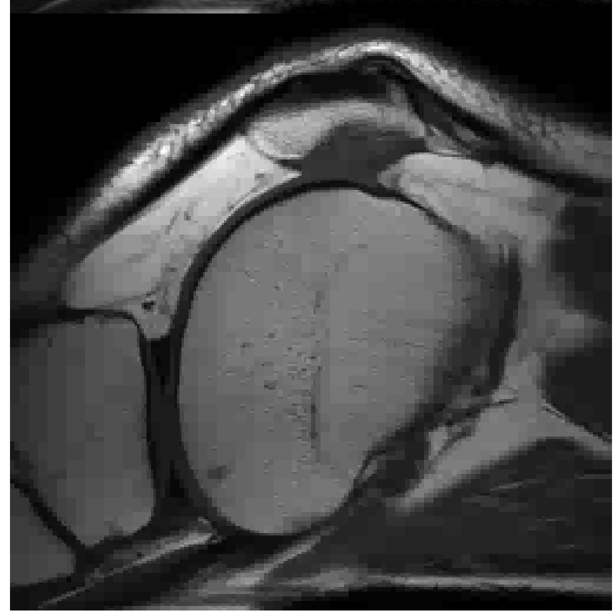
knee

original



original

14%
Haar wavelets



10%
Haar wavelets

Reconstruction SNR

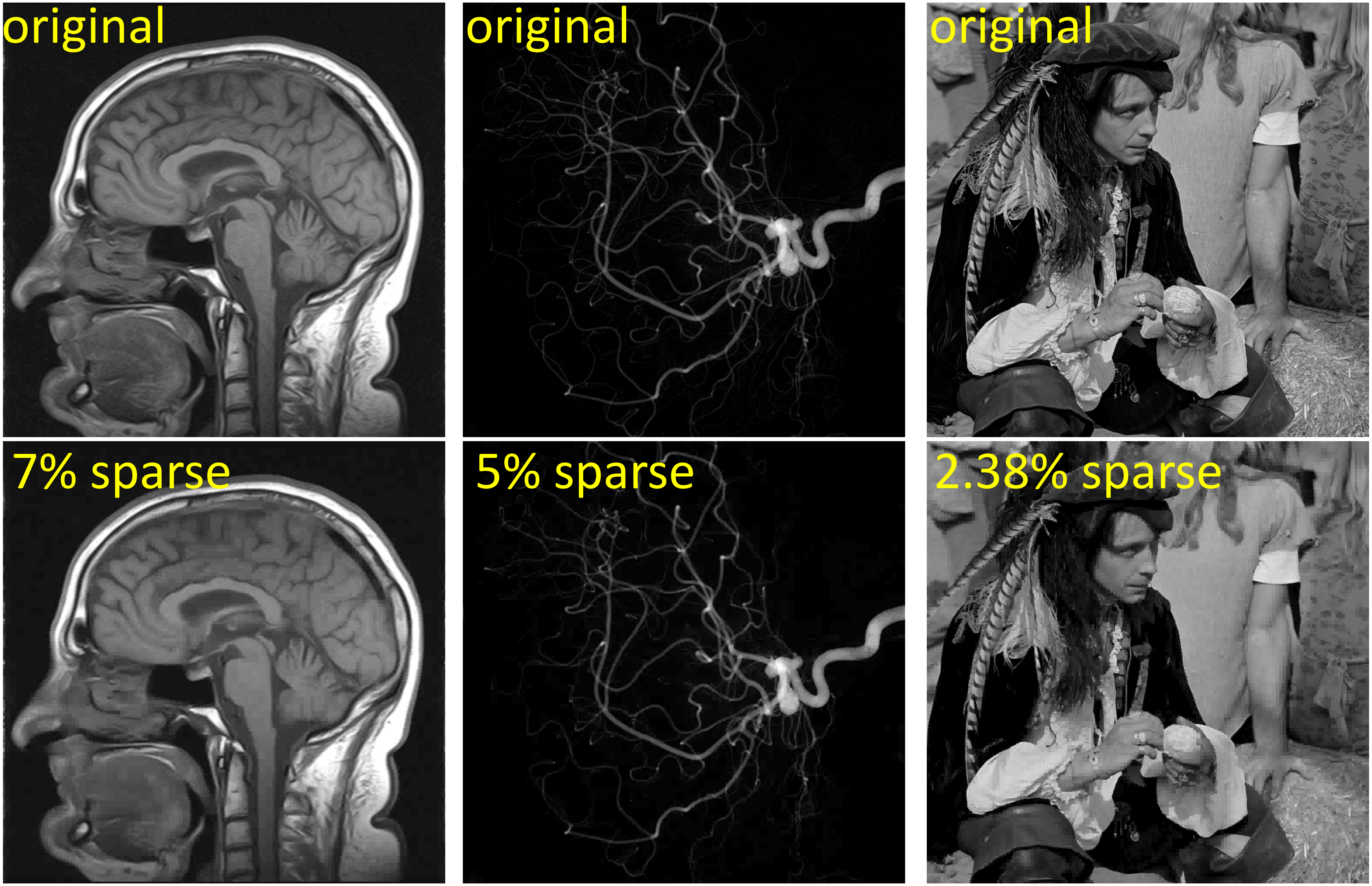
image	sparsity k	n/k	chirp	RM
cameraman	14%	1.78	109 dB	44 dB
knee	10%	2.5	119 dB	108 dB

$n : N = 1 : 4$

$n/N = 25\%$

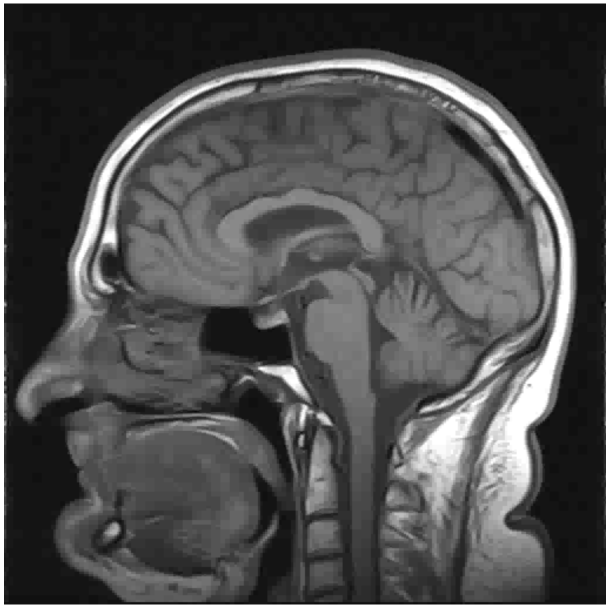
$$\Phi = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix}$$

brain 512 X 512 | vessel 512 X 512 | man 1024 X 1024



Result 1: 512 X 512 Medical Image

x_{actual}



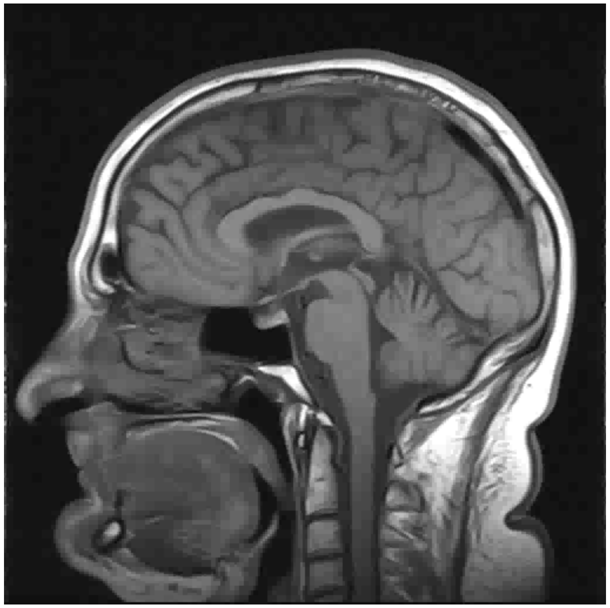
7% sparse

$x_{\text{reconstructed}}$

noiselets*+ ℓ_1**

SNR = 25 dB

$x_{\text{reconstructed}}$



chirp

SNR = 120 dB

$n : N = 1 : 4$

$n/N = 25\%$

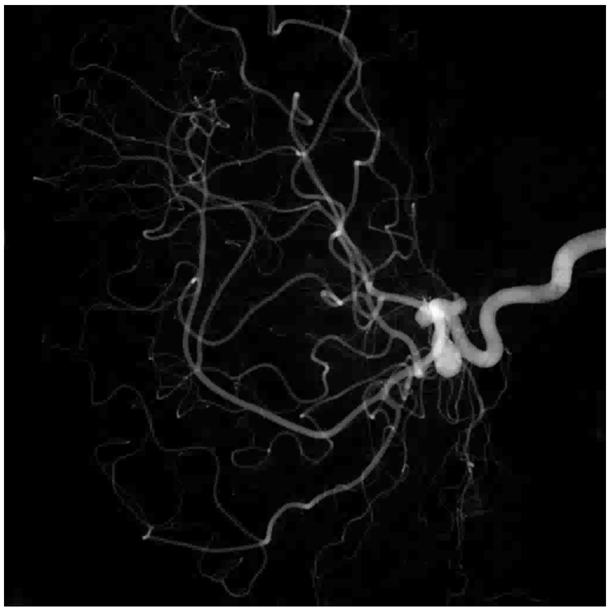
$$\Phi = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix}$$

**Zhang, Yang, and Yin, YALL1: Your ALgorithms for L1

*Candes and Romberg, sparsity and incoherence in compressive sampling

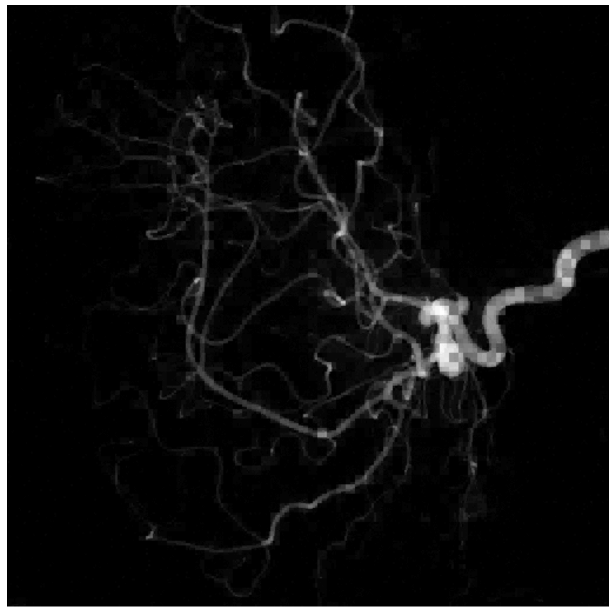
Result 2: 512 X 512 Medical Image

x_{actual}



5% sparse

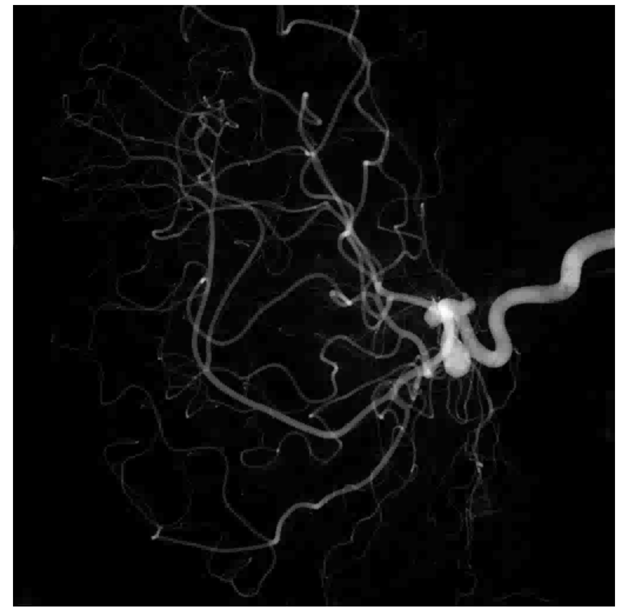
$x_{\text{reconstructed}}$



noiselets*+ ℓ_1**

SNR = 9.1 dB

$x_{\text{reconstructed}}$



chirp

SNR = 50 dB

$n : N = 1 : 8$

$n/N = 12.5\%$

$$\Phi = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \end{bmatrix}$$

**Zhang, Yang, and Yin, YALL1: Your ALgorithms for L1

*Candes and Romberg, sparsity and incoherence in compressive sampling

Result 3: 1024 X 1024 real image

x_{actual}

2.38% sparse

$x_{\text{reconstructed}}$

noiselets*+ ℓ_1**

SNR = 4.4 dB

$x_{\text{reconstructed}}$

chirp

SNR = 112 dB

$n : N = 1 : 16$

$n/N = 6.25\%$

$$\Phi = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 & U_{10} & U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} \end{bmatrix}$$

**Zhang, Yang, and Yin, YALL1: Your ALgorithms for L1

*Candes and Romberg, sparsity and incoherence in compressive sampling

Results

n/N	Image sparsity	n/k	**noiselets**	**chirp**	**RM**
25% $\Phi = [U_1 \ \ U_2 \ \ U_3 \ \ U_4]$	Brain 7%	3.6	25 dB	120 dB	116 dB
12.5% $\Phi = [U_1 \ \ U_2 \ \ U_3 \ \ U_4 \ \ U_5 \ \ U_6 \ \ U_7 \ \ U_8]$	Vessel 5%	2.5	9.1 dB	50 dB	10 dB
6.25% $\Phi = [U_1 \ \ U_2 \ \ U_3 \ \ U_4 \ \ U_5 \ \ U_6 \ \ U_7 \ \ U_8 \ \ U_9 \ \ U_{10} \ \ U_{11} \ \ U_{12} \ \ U_{13} \ \ U_{14} \ \ U_{15} \ \ U_{16}]$	Man 2.38%	2.6	4.4 dB	112 dB	109 dB

Stability of Algorithm

$$y = \Phi x_k$$

---

1. $$y = \Phi x_k + \mu$$

2. $$y = \Phi x + \mu$$

$\mu :$ noise

$x_k :$ k - sparse

$x :$ compressible

1. $y = \Phi x_k + \mu$

resilient to noise

image, size, sparsity	stan. dev. σ		noiselets	Chirp	RM
brain 512×512, 7%	0 0.01 0.05 0.1 0.2		29.7 24.7 16.8 12.6 8.1	127.0 46.4 27.5 22.0 15.5	121.6 45.6 26.8 21.4 14.9
vessel 512×512, 5%	0 0.01 0.05 0.1 0.2		37.1 16.4 7.5 2.8 -2.6	126.2 32.8 14.6 10.9 5.2	121.0 32.7 15.0 10.0 4.5
man 1024×1024, 2.38%	0 0.01 0.05 0.1 0.2		43.1 31.4 18.3 13.5 8.6	122.6 46.9 27.0 22.0 16.0	118.1 46.5 26.0 21.0 15.5

2. $y = \Phi x + \mu$

works for compressible signals

image, size	stan. dev. σ		noiselets	Chirp	RM
brain 512×512	0 0.01 0.05 0.1 0.2		23.4 22.4 16.5 12.5 8.0	28.4 28.0 25.2 21.3 15.3	25.7 25.7 24.9 20.9 14.7
vessel 512×512	0 0.01 0.05 0.1 0.2		12.0 11.5 6.9 2.6 -2.6	14.1 13.8 12.4 9.9 4.9	13.4 13.4 12.3 9.1 4.3
man 1024×1024	0 0.01 0.05 0.1 0.2		20.0 19.6 16.2 12.7 8.4	23.2 23.2 22.5 20.2 15.6	22.6 22.5 21.7 19.4 15.1

Conclusion

- New reconstruction algorithm by deterministic CS method especially suitable for images

- Initial best approximation method
  - Speed up solution
  - Decrease reconstruction error

- Better computational complexity for finding nonzero locations

- A fast matrix-vector multiplication via FFT drastically improves the least-squares solution time

- **Extend utility of CS with chirps and RM into regime of less sparsity**
- **Support imaging applications**

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