# The computation of stable and accurate Fourier extensions of smooth functions

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# Outline of the talk

Introduction

Fourier extensions

Convergence of Fourier extensions

Resolution power

Numerical stability

Fourier extensions based on equispaced data

# Fourier series

Let $f : [-1, 1] \to \mathbb{R}$. Its $N^{\mathrm{th}}$ partial Fourier series is

$$f_N(x) = \sum_{|n| \le N} \hat{f}_n \mathrm{e}^{\mathrm{i} n \pi x}, \qquad N \in \mathbb{N},$$

where

$$\hat{f}_n = \frac{1}{2} \int_{-1}^{1} f(x) \mathrm{e}^{-\mathrm{i} n \pi x} \, \mathrm{d}x, \quad n \in \mathbb{Z},$$

are the Fourier coefficients of $f$.

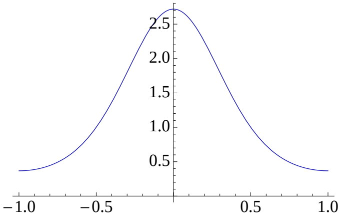
Fourier series are extremely effective tools in computations.

# Reason 1: rapid convergence of Fourier series

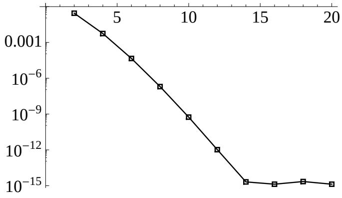
The Fourier series $f_N$ converges exponentially fast whenever $f$ is analytic and periodic, i.e.

$$\|f - f_N\|_\infty := \sup_{x \in [-1,1]} |f(x) - f_N(x)| \sim \rho^{-N},$$

for some $\rho > 1$.




$f(x) = \mathrm{e}^{\cos \pi x}$




$\|f - f_N\|_\infty$

# Reason 2: solving PDEs with Fourier series

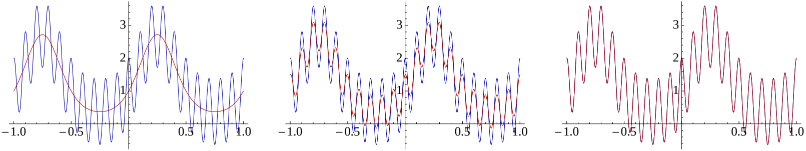
1. Fourier series lead to stable numerical algorithms (spectral methods) for PDEs.

2. Computations can be carried out rapidly, in $\mathcal{O}(N \log N)$ time, with the FFT.

# Reason 3: resolution power of Fourier series

Fourier series are good at resolving periodic oscillations.

- Obtain the optimal resolution constant of 2 d.o.f. per wavelength.




$N = 10$ $(\|f - f_N\|_\infty \approx 1)$ | $N = 20$ $(\|f - f_N\|_\infty \approx 10^{-1})$ | $N = 40$ $(\|f - f_N\|_\infty \approx 10^{-14})$

Graphs of $f(x) = \cos 20\pi x + \exp(\sin 2\pi x)$ (blue) and $f_N(x)$ (red).

Conversely, expansions in orthogonal polynomials (e.g. Chebyshev polynomials) have a higher resolution constant equal to $\pi$.

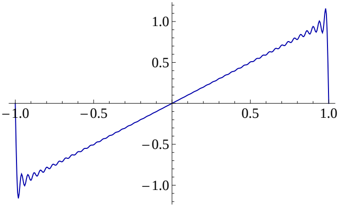
# Limitations of Fourier series I

Most functions are not periodic.

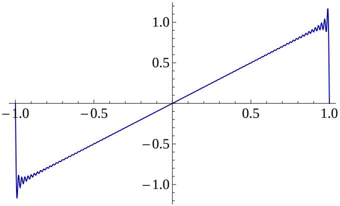
The Fourier series of a nonperiodic function gives a very poor approximation.

- Gibbs phenomenon.
- No uniform convergence.

E.g. $f(x) = x$:




$N = 50$




$N = 100$

# Limitations of Fourier series II

Fourier series are limited to simple geometries.

- E.g. intervals, (hyper)rectangles, parallelopipeds.

- Some extensions to certain triangles and simplices. But require rather unphysical notions of periodicity.

# Main question

Is there a way to retain the good properties of Fourier series of periodic functions, i.e.

(i) rapid convergence,

(ii) good resolution power,

(iii) easy manipulation via the FFT,

for nonperiodic functions, and functions defined in arbitrary domains?

# Answer

Yes! One can compute approximations of analytic, nonperiodic functions which

(i) are expressed in terms of a Fourier series,

(ii) converge exponentially fast,

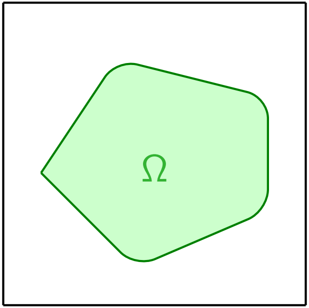
(iii) have a resolution constant that can be made arbitrarily close to 2 by an appropriate (function-independent) choice of a certain parameter,

(iv) are numerically stable.

The method is based on computing so-called Fourier extensions.

# An (old) idea

Seek to approximate a function $f : \Omega \to \mathbb{R}$ by a Fourier series on a larger, (hyper)rectangular domain.




Known as the Fourier extension problem.

# The Fourier extension problem

Existence/construction of extensions:

- Whitney (1934), Hestenes (1941), Fefferman (2005),...
- However, typically cannot obtain exponential convergence this way – no analytic and periodic extension of an arbitrary analytic function.
- Throughout, we shall never explicitly calculate extensions.

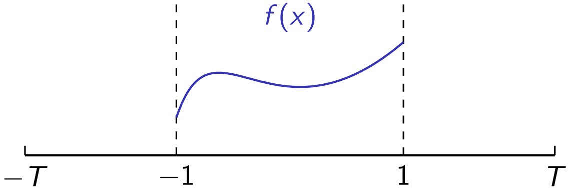
Computation of extensions:

- Boyd (2002), Bruno (2003), Bruno et al (2007), Huybrechs (2010), BA & Huybrechs (2011).
- Obtain exponential convergence, but only in the original domain $\Omega$.

Applications of extensions:

- Solution of PDEs in complex geometries, Bruno & Lyon (2010, 2011), Albin & Bruno (2011).

# One-dimensional Fourier extensions




We seek an approximation $f_N \in \mathcal{S}_N$, where

$$\mathcal{S}_N = \mathrm{span}\left\{ \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\frac{n\pi}{T}x} : n = -N, \ldots, N \right\},$$

is the set of Fourier series of degree $N$ on $[-T, T]$.

Question: how do we compute $f_N$?

# Computing $f_N$ – least squares

Define

$$f_N := \operatorname*{argmin}_{\phi \in \mathcal{S}_N} \|f - \phi\|, \qquad (\star)$$

where $\|g\|^2 = \int_{-1}^{1} |g(x)|^2 \, \mathrm{d}x$.

- Results in a linear system for the coefficients of $f_N$.
- We refer to $(\star)$ as the exact Fourier extension of $f$.

Problem: we need to know the integrals $\int_{-1}^{1} f(x) \mathrm{e}^{-\mathrm{i} \frac{n\pi}{T} x} \, \mathrm{d}x$.

# Computing $f_N$ – discrete least squares

Intstead, we can replace integrals by a quadrature, leading to

$$f_N := \operatorname*{argmin}_{\phi \in \mathcal{S}_N} \sum_{|n| \leq N} |f(x_n) - \phi(x_n)|^2. \qquad (\star)$$

- We refer to $(\star)$ as the discrete Fourier extension of $f$.

Question: what are good nodes to choose?

Answer: If $c(T) = \cos \frac{\pi}{T}$, let

$$x_n = \frac{T}{\pi} \cos^{-1} \left\{ \frac{1}{2}(1 - c(T)) \cos \left[ \frac{(2n+1)\pi}{2N+2} \right] + \frac{1}{2}(1 + c(T)) \right\},$$

for $n = 0, \ldots, N$, and define $x_{-n} = -x_n$.

The discrete and exact Fourier extensions are nearly identical.

- From now, assume that $f_N$ is computed via $(\star)$.

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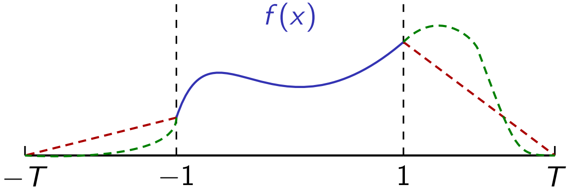
# Existence of extensions

Since

$$\|f - f_N\| \leq \|f - \phi\|, \quad \forall \phi \in \mathcal{S}_N,$$

fast convergence of $f_N \Leftrightarrow$ existence of periodic extensions of $f$ with arbitrary smoothness.




**Lemma**

*Let $f \in \mathrm{H}^k(-1,1)$, $k \in \mathbb{N}$. Then there exists a $\tilde{f} \in \mathrm{H}_0^k(-T,T)$ satisfying (i) $\tilde{f}|_{[-1,1]} = f$ and (ii) $\|\tilde{f}\|_{\mathrm{H}^k(-T,T)} \leq c\|f\|_{\mathrm{H}^k(-1,1)}$.*

# Algebraic/spectral convergence

**Theorem**

*Let* $f \in \mathrm{H}^k(-1,1)$*,* $k \in \mathbb{N}$*. Then*

$$\|f - f_N\| \leq c_k \left(\frac{N\pi}{T}\right)^{-k} \|f\|_{\mathrm{H}^k(-1,1)}.$$

- In other words, convergence at precisely the same rate as a Fourier series of a periodic function $g \in \mathrm{H}^k(-T,T)$.

Question: what if $f$ in analytic in [-1,1]?

# Exponential convergence: the key idea

Note that $f_N$ consists of the functions

$$\cos \tfrac{k\pi}{T} x, \quad \sin \tfrac{(k+1)\pi}{T} x, \qquad k = 0, \ldots, N.$$

If $c(T) = \cos \frac{\pi}{T}$ and

$$y = y(x) := \cos \frac{\pi}{T} x, \qquad y : [0,1] \to [c(T), 1],$$

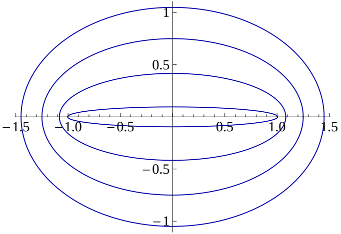
then

$$\cos \tfrac{k\pi}{T} x = T_k(y) \in \mathbb{P}_k, \qquad \sin \tfrac{(k+1)\pi}{T} x / \sin \tfrac{\pi}{T} x = U_k(y) \in \mathbb{P}_k.$$

Thus, $f_N$ is a sum of two polynomials expansions of degree $N$ in $y$, corresponding to the even and odd parts of $f$ respectively.

# Expansions in orthogonal polynomials

The expansion of an analytic function $g$ in (almost) any orthogonal polynomial system converges exponentially fast at rate $\rho$, where $\rho = \rho_{\mathsf{max}}$ corresponds to the largest Bernstein ellipse $B(\rho)$ within which $g$ is analytic.




$B(\rho)$, $\rho = 1.1, 1.5, 2, 2.5$

$$B(\rho) = \left\{ \frac{1}{2}\left(\rho e^{i\theta} + \rho^{-1} e^{-i\theta}\right) : \theta \in [-\pi, \pi) \right\}, \quad \rho \geq 1.$$

# Theorem on exponential convergence

**Theorem**

*Let $D(\rho)$, $\rho > 1$, be the image in the complex $x$-plane under the mapping $y = \cos \frac{\pi}{T} x$ of the Bernstein ellipse $B(\rho)$ in the complex $y$-plane centred around $[c(T), 1]$. Suppose that $f$ is analytic in $D(\rho^*)$ and not analytic in any $D(\rho')$ with $\rho' > \rho^*$. Then*

$$\|f - f_N\| \leq c_f \rho^{-N},$$

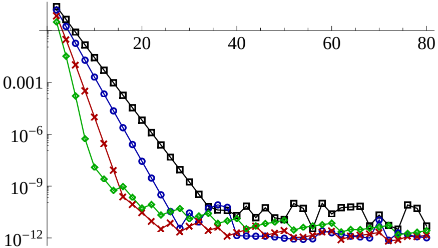
*where $\rho = \min\{\rho^*, E(T)\}$, and*

$$E(T) = \cot^2\left(\frac{\pi}{4T}\right).$$

- The map $y = \cos \frac{\pi}{T} x$ introduces a square-root type singularity at $y = -1$. This limits the maximal $\rho$ to $E(T)$.

# Numerical example

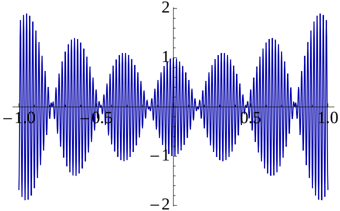
Let $T = \frac{4}{3}, \frac{3}{2}, 2, 4$:



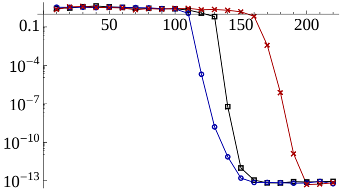

$\|f - f_N\|$ for $f(x) = \mathrm{e}^{5x}$

# Numerical comparison of resolution power

Consider $f(x) = (1 + x^2)\cos 10x \cos 100\pi x$.




$f(x)$




$\|f - f_N\|$

Here $f_N$ represent the Chebyshev series of $f$ (red), and the Fourier extension with $T = \frac{4}{3}$ (black) and $T = \frac{8}{7}$ (blue).

# The main result

By analysing the behaviour of the Fourier extension of

$$f(x) = \mathrm{e}^{\mathrm{i}\pi\omega x}, \quad x \in [-1, 1],$$

for large $\omega \gg 1$, we obtain:

**Theorem**

*The resolution constant $r = r(T)$ satisfies*

$$r(T) \leq 2T \sin\left(\frac{\pi}{2T}\right), \quad T > 1.$$

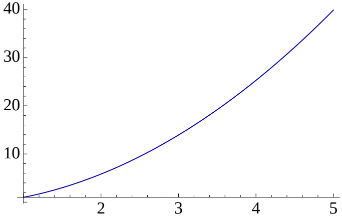
*In particular,*

$$r(T) \sim 2 + \mathcal{O}(T-1), \quad T \to 1,$$
$$r(T) \sim \pi + \mathcal{O}\left(T^{-2}\right), \quad T \to \infty.$$

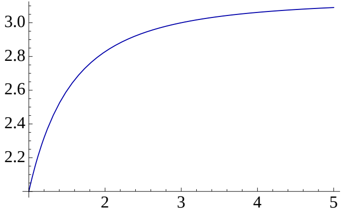
- Thus, the Fourier and polynomial resolution constants are the limiting values for $r(T)$.

# Resolution vs convergence

The parameter $T$ can be chosen by the user.




$E(T) = \cot^2\left(\frac{\pi}{4T}\right)$




$r(T) = 2T\sin\left(\frac{\pi}{2T}\right)$

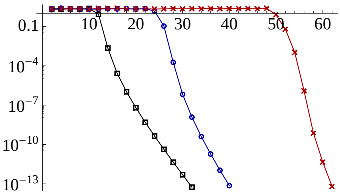
Large $T$: faster exponential convergence, but worse resolution

Small $T$: better resolution, but slower exponential convergence.

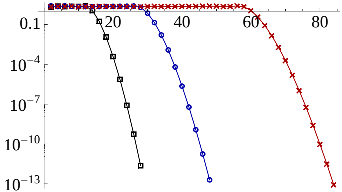
Varying $T$: (i.e. $T = 1 + N^{-\alpha}$, $0 < \alpha < 1$) formally optimal resolution $r(T) \to 2$, but much slower convergence $\sim \mathrm{e}^{-\pi N^{1-\alpha}}$.

# Numerical results – high precision

Using (very) high precision one sees:




$T = \sqrt{2}$



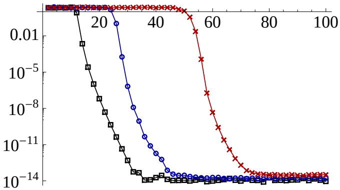

$T = 5$

The error $\|f - f_N\|$ for $f(x) = \mathrm{e}^{\mathrm{i}\pi\omega x}$ and $\omega = 10$ (black), $\omega = 20$ (blue), $\omega = 40$ (red).

| | $\omega = 10$ | $\omega = 20$ | $\omega = 40$ |
|---|---|---|---|
| $\frac{1}{2}\omega r(T)$ $(T = \sqrt{2})$ | 12.7 | 25.3 | 50.7 |
| $\frac{1}{2}\omega r(T)$ $(T = 5)$ | 15.5 | 30.9 | 61.8 |

# Numerical results – standard precision

However, in standard precision:

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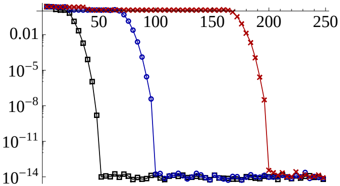


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- For large $T$, one sees much worse resolution in finite precision.

Either:

(i) The theorem is wrong!

(ii) The code has a bug!

(iii) The numerical solver does not do what we expect.

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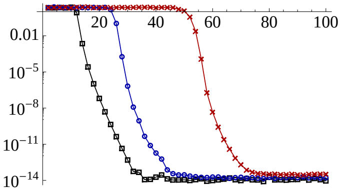


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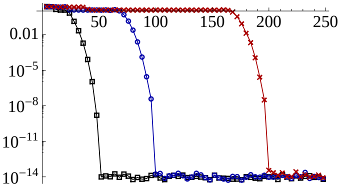


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# Frames

Let $\mathrm{H}$ be a Hilbert space. A set $\{\phi_n\}_{n\in\mathbb{Z}} \subset \mathrm{H}$ is a frame for $\mathrm{H}$ if

(i) $\{\phi_n\}_{n\in\mathbb{Z}}$ is dense in $\mathrm{H}$,

(ii) there exist $c_1, c_2 > 0$ such that

$$c_1\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, \phi_n\rangle|^2 \leq c_2\|f\|^2, \quad \forall f \in \mathrm{H}.$$

Frames are typically redundant. Any $f \in \mathrm{H}$ can have infinitely many representations of the form

$$f = \sum_{n\in\mathbb{Z}} \alpha_n\phi_n, \qquad \{\alpha_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}).$$

Redundancy implies ill-conditioning in numerical methods.

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# Fourier extensions

The set

$$\left\{ \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\frac{n\pi}{T}x} : n \in \mathbb{Z} \right\},$$

is a tight frame for $\mathrm{L}^2(-1,1)$.

- Redundancy: any $f \in \mathrm{L}^2(-1,1)$ has infinitely many extensions.

**Lemma**

*The condition number $\kappa_N$ of the discrete Fourier extension satisfies*

$$\kappa_N \sim E(T)^N.$$

- i.e. Fourier extensions are intrinsically unstable.

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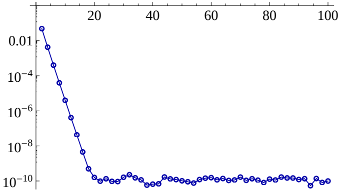
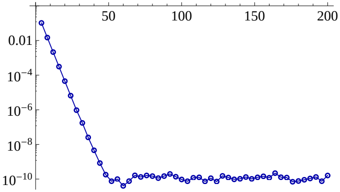
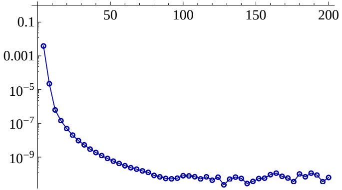
# A pleasant surprise




$f(x) = x$

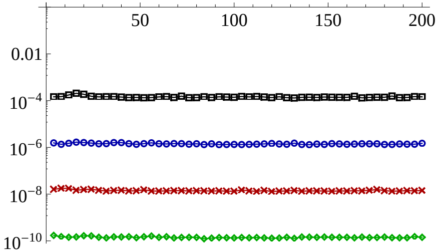



$f(x) = \frac{1}{1+25x^2}$




$f(x) = |x|^5$

# A pleasant surprise




Stability as a function of $N = 1, \ldots, 200$ for $\epsilon = 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}$.

# An underdetermined linear system

We need to solve the linear system $Aa = b$, where

$$A = \{\langle \phi_n, \phi_m \rangle\}_{|n|,|m| \leq N}\,, \qquad \phi_n(x) = \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\frac{n\pi}{T}x}.$$

Large $N$: redundancy $\Rightarrow$ columns of $A$ are near linearly dependent.

- Thus, $Aa = b$ becomes undetermined as $N \to \infty$.

Any solver will seek approximate solutions $a$ with small norm.

Question: how large can the coefficients of $f_N$ be?

# Magnitudes of the coefficients

**Lemma**

*Let $f_N = \sum_{|n|\leq N} a_n \phi_n$ and suppose that $\rho^*$ is the largest $\rho > 1$ such that $f$ is analytic within $D(\rho)$. Then*

$$\|a\| \leq c_f \begin{cases} \left(\frac{E(T)}{\rho^*}\right)^N & \rho^* < E(T), \\ \log N & \rho^* \geq E(T). \end{cases}$$

- $f$ analytic $\rho^* \geq E(T)$: $\|a\|$ very slowly growing.
- $f$ analytic $\rho^* < E(T)$ : $\|a\|$ exponentially large.
- $f$ not analytic : $\|a\|$ exponentially large.

If $f$ is not 'sufficiently' analytic, then $f_N$ will not be obtained from a numerical solver for large $N$.

# If not $f_N$, then what?

**Lemma**

*Let $f \in \mathrm{H}^k(-1,1)$. Then there exist $a^{[N]}$, $N \in \mathbb{N}$ such that*

$$\|a^{[N]}\| \leq c_k \|f\|_{\mathrm{H}^k(-1,1)}, \quad N \in \mathbb{N},$$

$$\|Aa^{[N]} - b\| \leq c_k \left(\frac{N\pi}{T}\right)^{-k} \|f\|_{\mathrm{H}^k(-1,1)}, \quad N \in \mathbb{N},$$

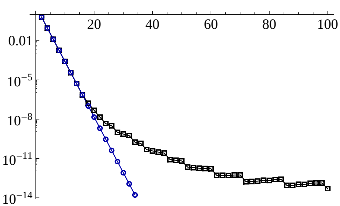
*and*

$$\left\| f - \sum_{|n| \leq N} a_n^{[N]} \phi_n \right\| \leq c_k \left(\frac{N\pi}{T}\right)^{-k} \|f\|_{\mathrm{H}^k(-1,1)}, \quad N \in \mathbb{N}.$$

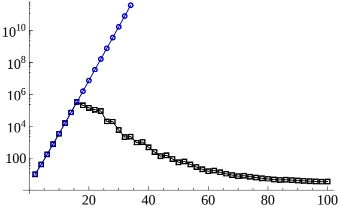
In other words, exponential convergence may be sacrificed for (high-order) algebraic convergence to retain numerical stability.

# Numerical example

Let $f(x) = \frac{1}{x - \frac{8}{7}}$.




$\|f - f_N\|$




$\|a\|$

The quantities $\|f - f_N\|$ and $\|a\|$ where $f_N$ is the 'theoretical' exact extension (blue) and the numerical solution (black).

Fourier extensions are intrinsically unstable, but numerically stable for all intents and purposes.

# Explanation of resolution power

When $T \gg 1$, theory predicts that $r(T) \approx \pi$. However, in finite precision we see $r(T) \approx 2T$.

**Lemma**

*Let $f(x) = \exp(\mathrm{i}\pi\omega x)$ and suppose that $f_N = \sum_{|n|\leq N} a_n \phi_n$ is the exact Fourier extension of $f$. Then, for $N < \frac{1}{2} r(T)\omega$, we have*

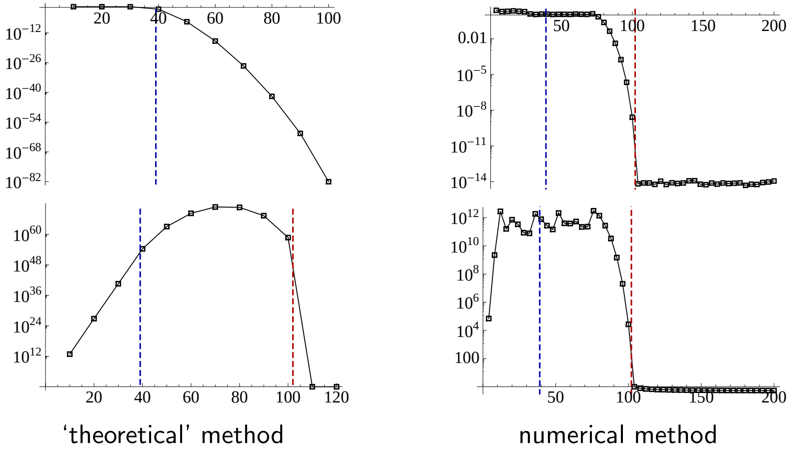
$$\|a\|_{l^2} \sim E(T)^N.$$

- The exact solution has large coefficients in the unresolved regime – in particular, when $T$ is large.
- For large $T$, $\omega$ cannot obtain the exact solution in finite precision.

# Numerical example

Consider $f(x) = \cos \omega \pi x$ with $\omega = \frac{51}{2}$, and $T = 4$.

- i.e. $\frac{1}{2}\omega r(T) \approx 39$, $\omega T \approx 100$.




Top: the error $\|f - f_N\|$. Bottom: the coefficient norm $\|a\|$.

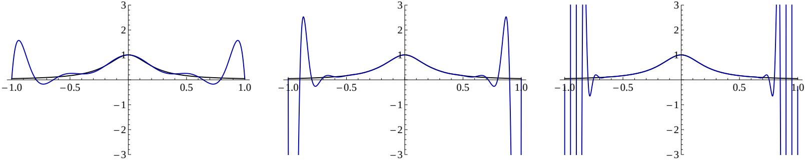
# Background

In many problems one faces equispaced data, i.e.

$$f\left(\tfrac{n}{N}\right), \quad |n| \leq N.$$

Equispaced data is difficult to handle.

Runge phenomenon: the polynomial interpolant of $f$ at equispaced nodes diverges unless $f$ is analytic in a sufficiently large region.



$N = 10$ $N = 20$ $N = 40$

Graphs of $f(x) = \frac{1}{1+20x^2}$ (black) and its equispaced polynomial interpolant (blue).

# A result of Platte, Trefethen & Kuijlaars

Problem: given $\{f(\frac{n}{N})\}_{|n|\leq N}$, recover $f$ to high accuracy.

Many methods have been proposed to do this. However,

**Theorem (Platte, Trefethen & Kuijlaars (2010))**

*"Any method that recovers all analytic functions f to exponential accuracy using only the grid values $\{f(\frac{n}{N})\}_{|n|\leq N}$ must be exponentially ill-conditioned."*

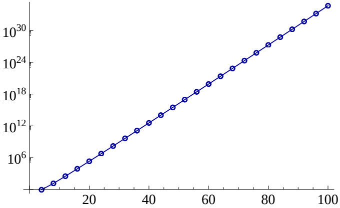
# Fourier extensions for equispaced data

Suppose that we define

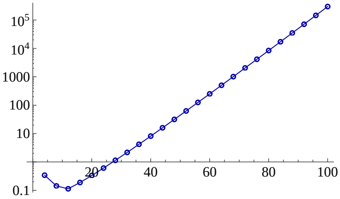
$$f_N := \underset{\phi \in \mathcal{S}_N}{\operatorname{argmin}} \sum_{|n| \leq M} |f(\tfrac{n}{M}) - \phi(\tfrac{n}{M})|^2,$$

where $M = \gamma N$ and $\gamma \geq 1$ a fixed oversampling parameter.

Example: $f(x) = \frac{1}{1+100x^2}$, in 'infinite' precision:




$\|f - f_N\|$ for $\gamma = 1$




$\|f - f_N\|$ for $\gamma = 2$

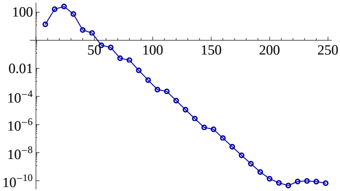
# Fourier extensions for equispaced data

Suppose that we define

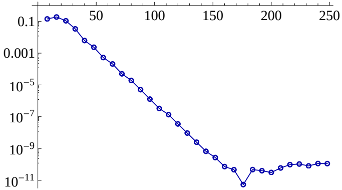
$$f_N := \underset{\phi \in \mathcal{S}_N}{\operatorname{argmin}} \sum_{|n| \leq M} \left| f\left(\tfrac{n}{M}\right) - \phi\left(\tfrac{n}{M}\right) \right|^2,$$

where $M = \gamma N$ and $\gamma \geq 1$ a fixed oversampling parameter.

Example: $f(x) = \frac{1}{1+100x^2}$, in standard precision:




$\|f - f_N\|$ for $\gamma = 1$




$\|f - f_N\|$ for $\gamma = 2$

# A theoretically unstable, divergent method

One can show that there is a Runge regime $D = D(\gamma, T)$ containing $[-1, 1]$ for which all functions with singularities inside $D$ will have a divergent Fourier extension $f_N$.

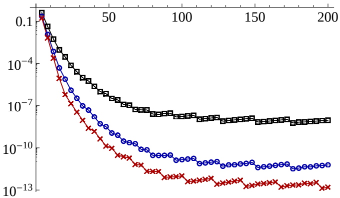
- Recall that $f_N$ is essentially a polynomial in $y = \cos \frac{\pi}{T} x$.
- The nodes $\cos\left(\frac{n\pi}{NT}\right)$ do not cluster quadratically near $y = c(T)$.

In fact, to guarantee theoretical convergence, one requires quadratic oversampling $M = \mathcal{O}\left(N^2\right)$.

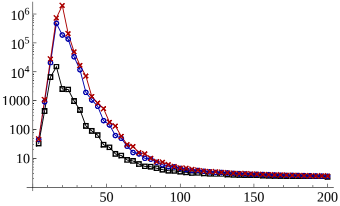
# A numerically stable, convergent method

However, Runge's phenomenon implies large coefficients! Thus, we never see divergence in finite precision.

Example: let $f(x) = \frac{1}{x - \frac{8}{7}}$, with $\gamma = 1$, $\gamma = 2$ and $\gamma = 4$.




$\|f - f_N\|$



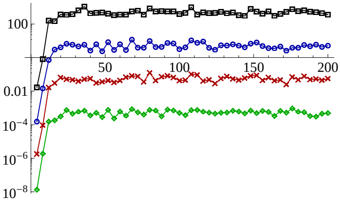

$\|a\|$

Exponential convergence is forfeited in some circumstances to retain stability.

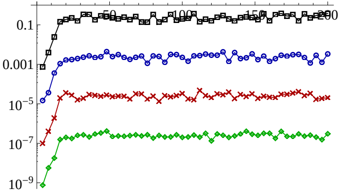
- No contradiction with Platte, Trefethen & Kuijlaars.

# Stability test

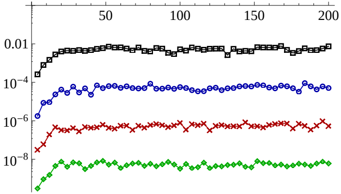
$\gamma = 1$




$\gamma = 2$




$\gamma = 4$




Stability as a function of $N = 1, \ldots, 200$ for $\epsilon = 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}$.

# Conclusion

One can compute rapidly convergent, numerically stable Fourier extensions of arbitrary smooth functions, even when only equispaced data is prescribed.

# References


## Convergence of Fourier extensions

- D. Huybrechs, *On the Fourier extension of non-periodic functions*. SIAM J. Numer. Anal., 47(6):4326–4355, 2010.

## Resolution power

- B. Adcock & D. Huybrechs, *On the resolution power of Fourier extensions for nonperiodic functions*. Submitted, 2011.

## Stability

- B. Adcock, D. Huybrechs & J. Martín–Vaquero, *On the stability of Fourier extensions*. In preparation, 2011.