# Generalized sampling
## A new framework for image and signal reconstruction

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# Outline of the talk

Introduction

Generalized sampling

Reconstructions from Fourier samples

Generalized sampling for nonuniform samples

Generalized sampling and infinite-dimensional compressed sensing

## Introduction

Generalized sampling

Reconstructions from Fourier samples

Generalized sampling for nonuniform samples

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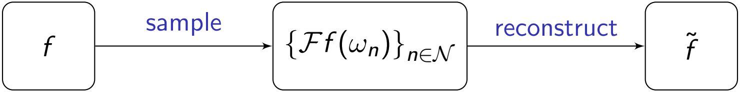
# Reconstructions from the Fourier transform

Fundamental problem: recover a image/signal $f$ from pointwise samples of its Fourier transform (FT)

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x) \mathrm{e}^{2\mathrm{i}\pi\omega\cdot x} \, \mathrm{d}x.$$

E.g. Magnetic Resonance Imaging (MRI).



# Key issues

1. The sampling scheme is fixed and cannot be altered.
2. Taking many samples is expensive/infeasible. Thus, one wants to reconstruct $f$ using as few samples as possible.
3. Sampling frequencies $\{\omega_n\}$ may be uniform or nonuniform.
4. Samples are always noisy. Also other effects, e.g jitter.

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## The Shannon sampling theorem

Let $f \in \mathrm{L}^2(\mathbb{R})$ with $\operatorname{supp}(f) \subseteq [-1,1]$. Then $f$ is determined uniquely by

$$\{\mathcal{F}f(n\epsilon)\}_{n\in\mathbb{Z}}, \qquad (\epsilon \leq \tfrac{1}{2}).$$

Specifically,

$$f(x) = \epsilon \sum_{n\in\mathbb{Z}} \mathcal{F}f(n\epsilon)\mathrm{e}^{2\pi \mathrm{i}\epsilon n x}.$$

However, in practice, we cannot access or process all samples of $f$. Thus, Shannon's Theorem gives rise to the approximation

$$f_N(x) = \epsilon \sum_{n=-N}^{N} \mathcal{F}f(n\epsilon)\mathrm{e}^{2\pi \mathrm{i}\epsilon n x}.$$

- In other words, the partial Fourier series of $f$.

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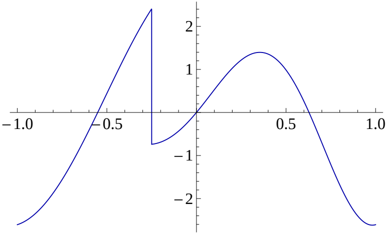
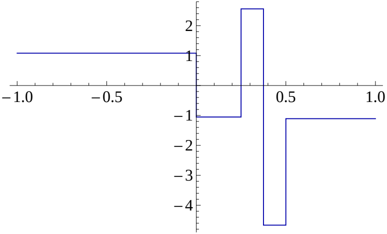
# Fourier series reconstructions

Let $\epsilon = \frac{1}{2}$, $N = 50$:




The function $f(x)$

# Fourier series reconstructions

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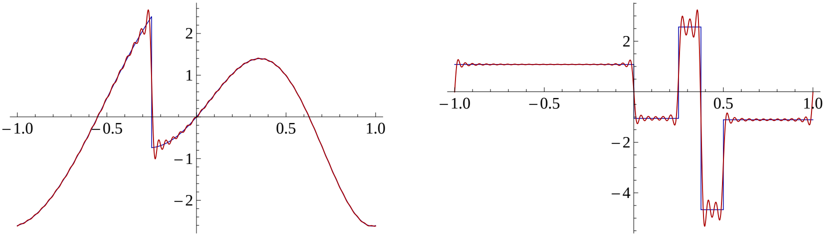


The functions $f(x)$ and $f_N(x)$

# Fourier series reconstructions

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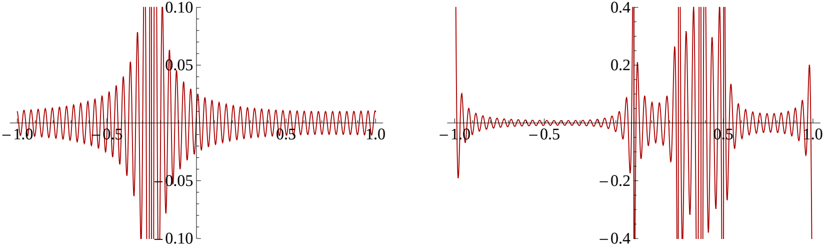
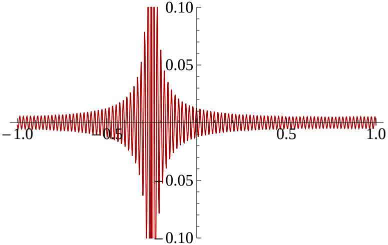
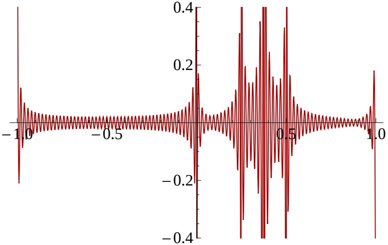



The error $f(x) - f_N(x)$

# Fourier series reconstructions

Let $\epsilon = \frac{1}{2}$, $N = 100$:

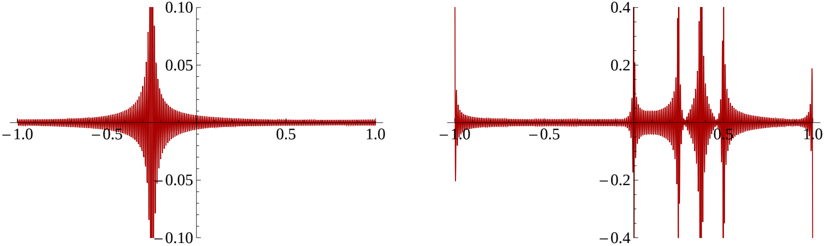



The error $f(x) - f_N(x)$

# Fourier series reconstructions

Let $\epsilon = \dfrac{1}{2}$, $N = 200$:



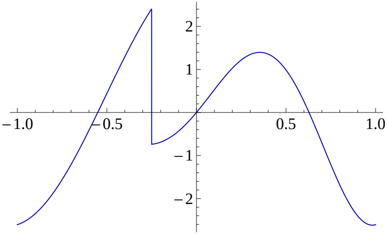

The error $f(x) - f_N(x)$

The coefficients $\mathcal{F}f(n\epsilon)$ decay very slowly as $|n| \to \infty$.

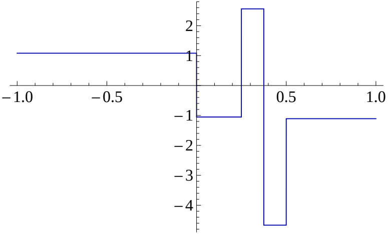
# Main question

Given the first $N$ Fourier samples $\{\mathcal{F}f(n\epsilon)\}_{n=-N}^{N}$, is there a better way to recover $f$ than the Fourier series $f_N$?

# Other ways to reconstruct $f$ from its samples




$f_1(x)$




$f_2(x)$

Both functions very poorly represented by $f_N$. However,

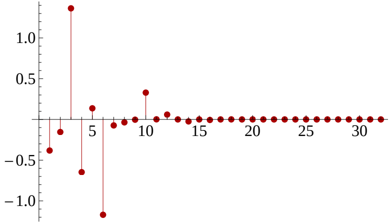
- $f_1$ is very well approximated by piecewise polynomials,
- $f_2$ is very well approximated by Haar wavelets.

# Other ways to reconstruct $f$ from its samples

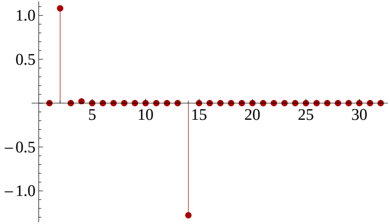
Write

$$f_i(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n, \quad i = 1, 2,$$

where $\{\phi_n\}_{n \in \mathbb{N}}$ are either piecewise polynomials or Haar wavelets.




coefficients $\{\alpha_n\}$ for $f_1$




coefficients $\{\alpha_n\}$ for $f_2$

- In either case, if we knew $\alpha_1, \ldots, \alpha_{32}$ we could recover $f_i$ with error $\approx 10^{-10}$.

# Main problem

More generally, let

$$\hat{f}_n, \quad n \in \mathbb{N},$$

be measurements of $f$ (e.g. samples of $\mathcal{F}f$).

Key assumption: suppose that we know that $f$ has a 'good' representation in a basis $\{\phi_n\}$, i.e. $\alpha_n \to 0$ rapidly.

Main problem:

Given the first $N$ measurements $\{\hat{f}_n\}_{n=1}^N$ recover the coefficients $\{\alpha_n\}$ in the basis $\{\phi_n\}$.

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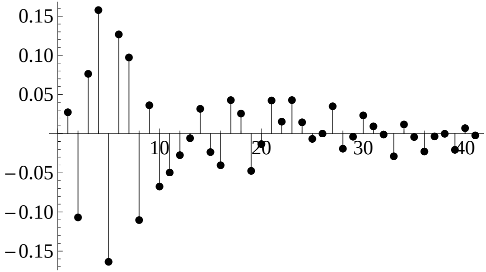
# Key consideration I: quasi-optimality

Suppose that we have a mapping

$$\mathcal{L} : \{\hat{f}_1, \ldots, \hat{f}_N\} \longmapsto \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M\}.$$




The coefficients $\alpha_1, \alpha_2, \ldots$.

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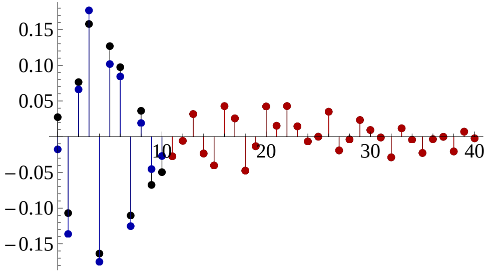


The coefficients $\alpha_1, \alpha_2, \ldots$ and approximate coefficients $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M$.

# Key consideration I: quasi-optimality

Suppose that we have a mapping

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Error equation:

$$f - \sum_{n=1}^{M} \tilde{\alpha}_n \phi_n = \sum_{n=1}^{M} (\alpha_n - \tilde{\alpha}_n)\phi_n + \sum_{n=M+1}^{\infty} \alpha_n \phi_n$$

total error — regularization error — truncation error

Its important that

regularization error $\approx$ truncation error, (quasi-optimality).

# Key consideration II: numerical stability

The mapping $\mathcal{L} : \{\hat{f}_1, \dots, \hat{f}_N\} \longmapsto \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_M\}$ should be numerically stable, i.e. the condition number

$$\|\mathcal{L}\|\|\mathcal{L}^{-1}\| \ll \infty,$$

to avoid large errors due to

- round-off error,
- noise,
- jitter,
- shock capturing.

# Hilbert space formulation

Let $\mathrm{H}$ be a separable Hilbert space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

- Let $\{\psi_n\}_{n\in\mathbb{N}}$ be an orthonormal sampling basis.
- Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal reconstruction basis.

E.g. Fourier sampling: $\mathrm{H} = \mathrm{L}^2(-1,1)$, $\psi_n(x) := \mathrm{e}^{2\pi \mathrm{i} \epsilon n x}$.

## The reconstruction problem

Given the first $N$ measurements

$$\hat{f}_n = \langle f, \psi_n \rangle, \quad n = 1, \ldots, N,$$

of $f \in \mathrm{H}$, compute the coefficients $\{\alpha_n\}_{n\in\mathbb{N}}$ of $f$ with respect to the reconstruction basis $\{\phi_n\}_{n\in\mathbb{N}}$.

# Key idea

Allow the parameters

- $N$ – the number of measurements,
- $M$ – the number of coefficients $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M$ to be computed,

to differ. Specifically, let $N > M$.

# Best possible reconstruction

The best reconstruction of $M$ coefficients is obviously

$$\tilde{\alpha}_m = \alpha_m = \langle f, \phi_m \rangle, \quad m = 1, \ldots, M.$$

The reconstruction

$$f_M = \sum_{m=1}^{M} \alpha_m \phi_m,$$

is the orthogonal projection of $f$ onto

$$\mathrm{T}_M = \mathrm{span}\{\phi_1, \ldots, \phi_M\} \subset \mathrm{H}, \quad \text{(reconstruction space)}.$$

Of course, we don't know $\{\alpha_m\}_{m=1}^{M}$. However, note that, by definition,

$$\langle f_M, \phi_m \rangle = \langle f, \phi_m \rangle, \quad m = 1, \ldots, M.$$

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# Generalized sampling

Define $\mathcal{P}_N : \mathrm{H} \to \mathrm{S}_N := \mathrm{span}\{\psi_1, \dots, \psi_N\}$ by

$$\mathcal{P}_N g = \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n.$$

Note: $\mathcal{P}_N$ is the orthogonal projection onto $\mathrm{S}_N$.

Generalized sampling: define $f_{N,M} = \sum_{m=1}^{M} \tilde{\alpha}_m \phi_m \in \mathrm{T}_M$ by

$$\langle \mathcal{P}_N f_{N,M}, \phi_m \rangle = \langle \mathcal{P}_N f, \phi_m \rangle, \quad n = 1, \dots, M.$$

- A linear system for $\tilde{\alpha}_1, \dots, \tilde{\alpha}_M$ involving only the given measurements $\hat{f}_1, \dots, \hat{f}_N$.

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# Intuitive explanation

Recall that

$$\langle f_M, \phi_m \rangle = \langle f, \phi_m \rangle, \quad m = 1, \ldots, M, \tag{1}$$

and

$$\langle \mathcal{P}_N f_{N,M}, \phi_m \rangle = \langle \mathcal{P}_N f, \phi_m \rangle, \quad n = 1, \ldots, M. \tag{2}$$

The operators $\mathcal{P}_N \to \mathcal{I}$ strongly on $\mathrm{H}$ as $N \to \infty$. Thus

$$f_{N,M} \approx f_M, \quad N \to \infty.$$

Hence for sufficiently large $N$, we expect 'good' behaviour of $f_{N,M}$.

# Main theorem

Let

$$C_{N,M} = \inf \{ \|\mathcal{P}_N \phi\| : \phi \in \mathrm{T}_M, \ \|\phi\| = 1 \}.$$

- Key point: for fixed $M$, $C_{N,M} \to 1$ as $N \to \infty$.

**Theorem (BA, Hansen)**

*For each $M \in \mathbb{N}$, there exists an $N_0 \in \mathbb{N}$ such that $f_{N,M}$ exists and is unique for all $N \geq N_0$, and satisfies the sharp bounds*

$$\|f - f_M\| \leq \|f - f_{N,M}\| \leq \frac{1}{C_{N,M}} \|f - f_M\|.$$

*Specifically, $N_0$ is the least $N$ such that $C_{N,M} > 0$.*

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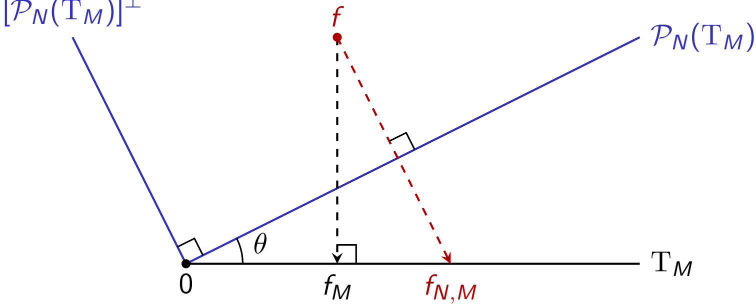
# Geometric interpretation

The map $f \mapsto f_{N,M}$ is precisely the oblique projection onto $\mathrm{T}_M$ along $[\mathcal{P}_N(\mathrm{T}_M)]^\perp$. Moreover,

$$C_{N,M} = \cos\theta,$$

where $\theta$ is the angle between the subspaces $\mathrm{T}_M$ and $\mathcal{P}_N(\mathrm{T}_M)$.




- $\mathrm{T}_M$ and $\mathcal{P}_N(\mathrm{T}_M)$ cannot be near-perpendicular for large $N$. Hence $f_{N,M}$ is well-defined, and $f_{N,M} \approx f_M$.

# Numerical implementation

The equations

$$\langle \mathcal{P}_N f_{N,M}, \phi_m \rangle = \langle \mathcal{P}_N f, \phi_m \rangle, \quad n = 1, \ldots, M,$$

are equivalent to a $N \times M$ linear least squares system for $\{\tilde{\alpha}_m\}_{m=1}^{M}$.

- One can also show that the condition number

$$\|\mathcal{L}\| \|\mathcal{L}^{-1}\| \leq \frac{1}{C_{N,M}},$$

where $\mathcal{L} : \{\hat{f}_1, \ldots, \hat{f}_N\} \mapsto \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M\}$.

- The total computational cost in computing $f_{N,M}$ is at worst

$$\mathcal{O}\left( \frac{1}{C_{N,M}} NM \right).$$

# The stable sampling rate

Define the stable sampling rate

$$\Theta(M;\theta) = \min\left\{N \in \mathbb{N} : C_{N,M} > \theta\right\}, \quad \theta \in (0,1).$$

For given $M$, setting $N \geq \Theta(M;\theta)$ ensures

1. Existence and uniqueness of $f_{N,M}$.
2. Numerical stability: $\|\mathcal{L}\|\|\mathcal{L}^{-1}\| \leq \frac{1}{\theta}$.
3. Quasi-optimality: $\|f - f_{N,M}\| \leq \frac{1}{\theta}\|f - f_M\|$.

Note:

- This is a fundamentally new viewpoint on reconstruction.
- $\Theta(M;\theta)$ is a realization of certain concepts in computational spectral theory concerning the computation of spectra of operators.

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# Behaviour of the stable sampling rate

$\Theta(M;\theta)$ can always be computed numerically. However, it is also vitally important to determine analytic bounds.

Examples of known bounds:

1. Fourier samples, Haar wavelets: $\Theta(M;\theta) = c_\theta M$.
2. Fourier samples, (piecewise) polynomials: $\Theta(M;\theta) = c_\theta M^2$.

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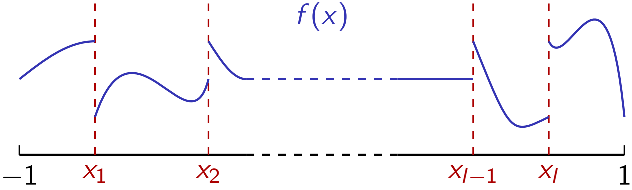
# Problem formulation




We seek to reconstruct $f$ in terms of piecewise orthogonal polynomials on $[-1, 1]$.

Note: in practice one needs to locate $x_1, \ldots, x_l$ to high accuracy.

- Known as edge detection, e.g. concentration kernel methods (Gelb, Tadmor, Tanner,...).
- Edge detection is an important source of errors. Any method must be robust w.r.t. such errors.

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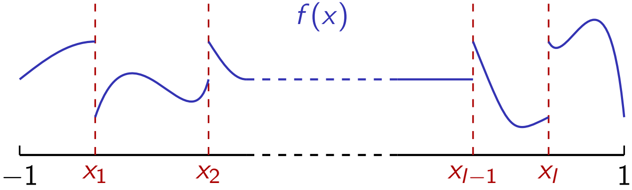


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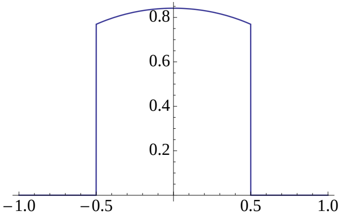
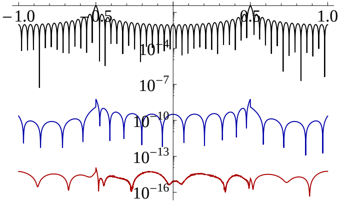
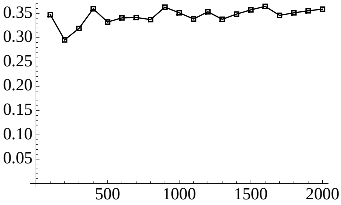
# Numerical example I



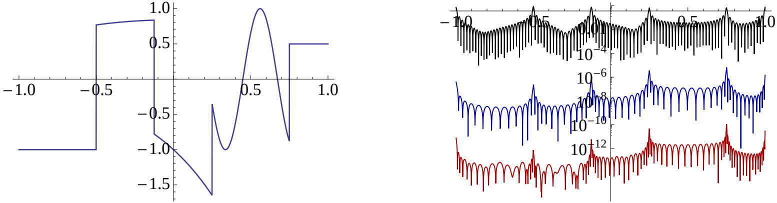




Left: $f(x) = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}(x)\sin(\cos x)$. Right: Fourier series (black), generalized sampling with $N = 25$, $M_0 = M_2 = \frac{1}{2}M_1 = 5$ (blue) and $N = 50$, $M_0 = M_2 = \frac{1}{2}M_1 = 7$ (red).

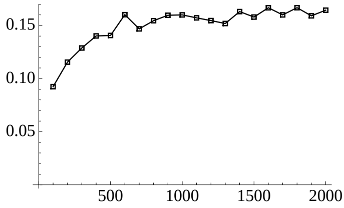



The quantity $C_{N,M}$ against $N$, where $M_0 = M_2 = \frac{1}{2}M_1 = \lceil\sqrt{N}\rceil$.

# Numerical example II

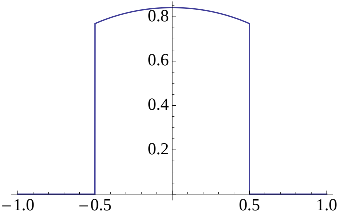


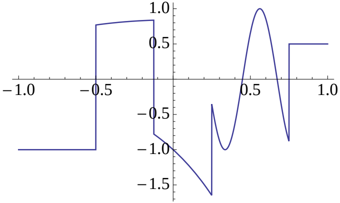

Left: $f(x)$. Right: Fourier series (black), generalized sampling with $N = 100$, $M_0 = \ldots = M_4 = 13$ (blue) and $N = 200$, $M_0 = \ldots = M_4 = 18$ (red).

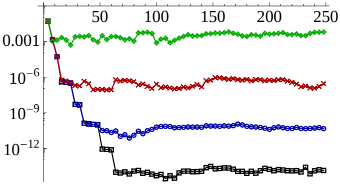



The quantity $C_{N,M}$ against $N$, where $M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{2} N} \rceil$.

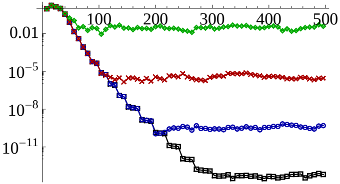
# Robustness I: noise



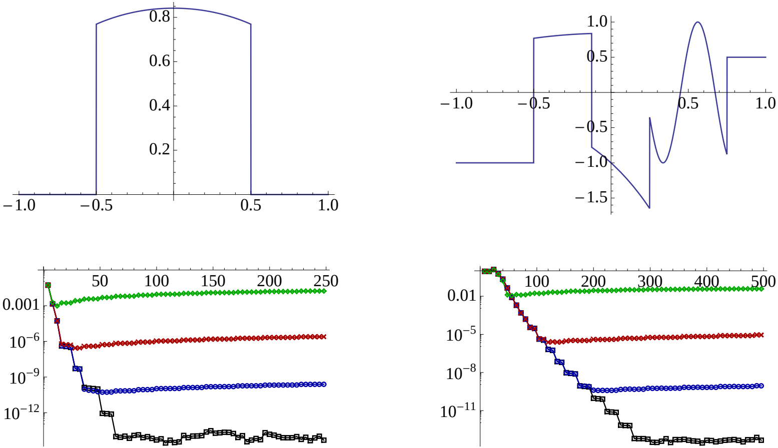

$$M_0 = M_2 = \frac{1}{2} M_1 = \lceil \sqrt{N} \rceil$$




$$M_0 = \ldots = M_4 = \left\lceil \sqrt{\tfrac{3}{2} N} \right\rceil$$

Top row: $f(x)$. Bottom row: the error $\|f - f_{N,M}\|$ against $N$ with noise at amplitudes $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$.

# Robustness II: edge detection errors

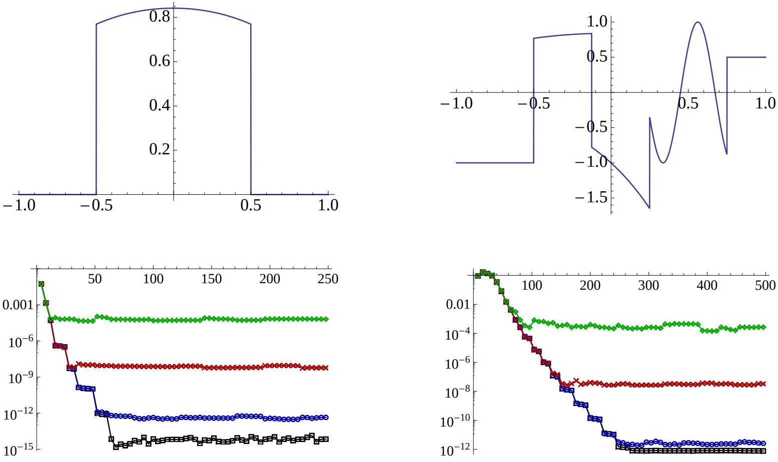



$M_0 = M_2 = \frac{1}{2} M_1 = \lceil \sqrt{N} \rceil$ $\qquad$ $M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{2} N} \rceil$

Top row: $f(x)$. Bottom row: the error $\|f - f_{N,M}\|$ against $N$ with edge detection errors of magnitude $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$.

- It can be shown that there is at worst linear drift in $M = \sqrt{N}$.

# Robustness III: jitter




$M_0 = M_2 = \frac{1}{2} M_1 = \lceil \sqrt{N} \rceil$ $\qquad$ $M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{2} N} \rceil$

Top row: $f(x)$. Bottom row: the error $\|f - f_{N,M}\|$ against $N$ with jitter errors of magnitude $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$.

- It can be shown that the best achievable accuracy scales like $\mathcal{O}(\epsilon)$.

Introduction

Generalized sampling

Reconstructions from Fourier samples

Generalized sampling for nonuniform samples

Generalized sampling and infinite-dimensional compressed sensing

# Motivation

In MRI one takes measurements

$$\{\mathcal{F}f(\omega_n)\}_{n=1}^N.$$

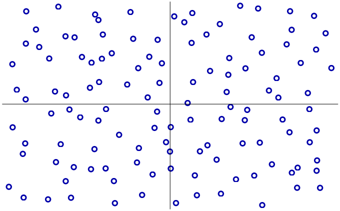
What types of sampling schemes $\{\omega_n\}_{n=1}^N$ are used in practice?

Equispaced:

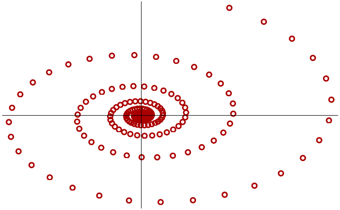
1D 2D

# Motivation

Jittered:



Logarithmic:



# Motivation

We need robust reconstruction algorithms that can handle (potentially highly) nonuniform sampling strategies.

Problem: the sampling system

$$\psi_n(x) = \mathrm{e}^{2\mathrm{i}\pi\omega_n\cdot x}, \quad n = 1, \ldots, N,$$

will not be orthonormal typically unless $\{\omega_n\}$ are equispaced.

# Frames

**Definition**

A system $\{\psi_n\}_{n\in\mathbb{N}}$ is a frame for a Hilbert space $\mathrm{H}$ if

- $\{\psi_n\}_{n\in\mathbb{N}}$ is dense in $\mathrm{H}$,
- there exist $c_1, c_2 > 0$ such that

$$c_1\|g\|^2 \leq \sum_{n=1}^{\infty} |\langle g, \psi_n\rangle|^2 \leq c_2\|g\|^2, \quad \forall g \in \mathrm{H}.$$

Note: the frame operator $\mathcal{P} : \mathrm{H} \to \mathrm{H}$, $\mathcal{P}g = \sum_{n=1}^{\infty}\langle g, \psi_n\rangle\psi_n$, is well-defined, bounded, self-adjoint, and we have

$$c_1\|g\|^2 \leq \langle \mathcal{P}g, g\rangle \leq c_2\|g\|^2.$$

# Generalized sampling with frames

Let

$$\mathcal{P}_N g = \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n,$$

and define

$$\langle \mathcal{P}_N f_{N,M}, \phi_m \rangle = \langle \mathcal{P}_N f, \phi_m \rangle, \quad m = 1, \ldots, M.$$

Note:

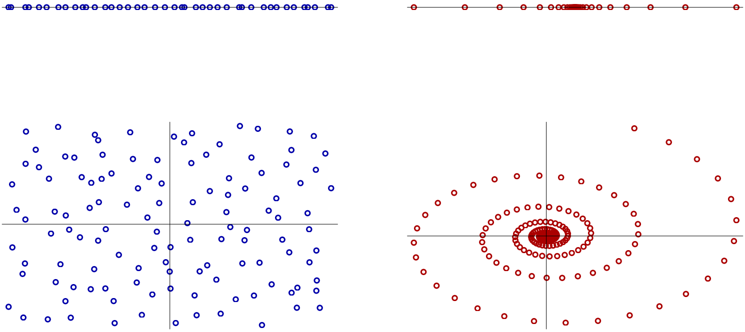
- $\mathcal{P}_N \to \mathcal{P}$ strongly on $\mathrm{H}$.
- $\langle \mathcal{P} \cdot, \cdot \rangle$ is an equivalent inner product on $\mathrm{H}$.

Hence, generalized sampling works equally well for frames as it does for orthonormal bases.

- In particular, whenever $\{\omega_n\}$ give rise to a Fourier frame then we may use generalized sampling.

# The non-frame case

Jittered sampling often gives rise to a Fourier frame. However, log-sampling need not.



jittered    logarithmic

- In general $\{\omega_n\}_{n=1}^{N}$ may also depend on $N$, i.e. $\{\omega_{n,N}\}_{n=1}^{N}$.

# Generalized sampling for (highly) nonuniform samples

Define the operator

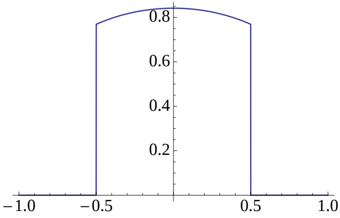
$$\mathcal{P}_N g(x) = \sum_{n=1}^{N} \mu_{n,N} \mathcal{F} g(\omega_{n,N}) \mathrm{e}^{2\mathrm{i}\pi\omega_{n,N} x}.$$

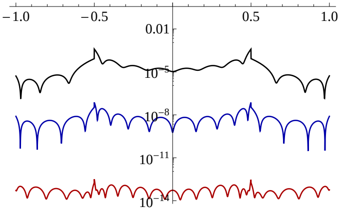
- $\mu_{n,N} = (\omega_{n+1,N} - \omega_{n,N})$ is a density compensation factor.
- Local clustering of $\{\omega_{n,N}\}$ is compensated by $\mu_{n,N}$.

Work in progress: what conditions on $\{\omega_{n,N}\}_{n=1}^{N}$ ensure that generalized sampling works for this (or a similar) choice of operator?

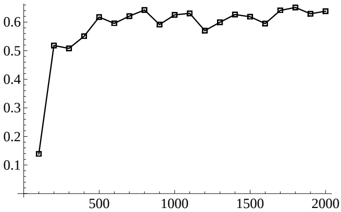
- Partial result: if $\delta^{[N]} := \max |\omega_{n+1,N} - \omega_{n,N}| \leq \delta < 1$, $\forall N \in \mathbb{N}$, then generalized sampling works with $\mathcal{P}_N$ as above.

# Numerical example I



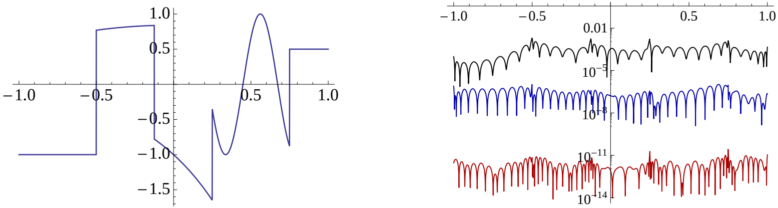




Generalized sampling with $N = 20$, $M_0 = M_2 = \frac{1}{2} M_1 = 5$ (black), $N = 40$, $M_0 = M_2 = \frac{1}{2} M_1 = 4$ (blue) and $N = 80$, $M_0 = M_2 = \frac{1}{2} M_1 = 7$ (red).

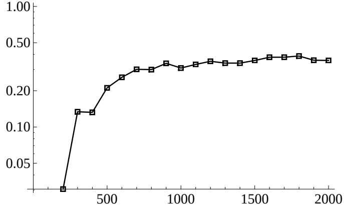



The quantity $C_{N,M}$ against $N$, where $M_0 = M_2 = \frac{1}{2} M_1 = \lceil \frac{1}{2} \sqrt{N} \rceil$.

# Numerical example II



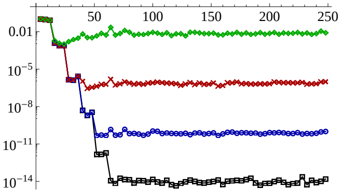

Generalized sampling with $N = 100$, $M_0 = \ldots = M_4 = 9$ (black), $N = 200$, $M_0 = \ldots = M_4 = 13$ (blue) and $N = 400$, $M_0 = \ldots = M_4 = 18$ (red).



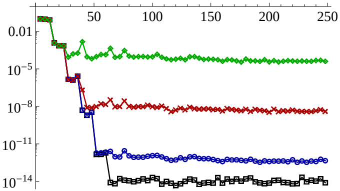

The quantity $C_{N,M}$ against $N$, where $M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{4} N} \rceil$.

# Robustness

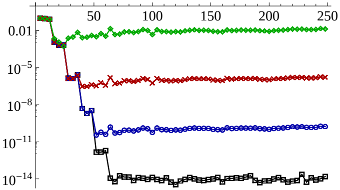
## Noise




## Jitter




## Edge detection



# Infinite-dimensional compressed sensing

Suppose $f$ is actually sparse in $\{\phi_n\}$, i.e.

$$|\{n : \alpha_n \neq 0\}| = k \ll \infty.$$

Question: can we recover $f$ exactly using $\mathcal{O}(k)$ measurements?

Answer: yes! By combining generalized sampling ideas with existing tools from finite-dimensional compressed sensing.

- Leads to an important generalization of compressed sensing to infinite-dimensional signal models.

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# A theorem on infinite-dimensional CS

## Theorem (BA,Hansen)

*Given*

$$f = \sum_{n=1}^{\infty} \alpha_n \phi_n, \qquad \Delta = \{m : \alpha_n \neq 0\},$$

*suppose that* $\Delta \subset \{1, \dots, M\}$ *for some* $M \in \mathbb{N}$. *Let* $\epsilon > 0$ *be arbitrary. Then, there exists an integer* $N \in \mathbb{N}$ *depending on* $M$ *and* $|\Delta|$ *only such that the following holds: if* $\Omega \subset \{1, \dots, N\}$, $|\Omega| = K$, *is chosen uniformly at random, then, with probability greater than* $1 - \epsilon$, $f$ *can be recovered exactly from the samples* $\{\hat{f}_m : m \in \Omega\}$ *given that* $K$ *is proportional to*

$$|\Delta| \cdot \log(\epsilon^{-1} + 1) \cdot \log(NM\sqrt{|\Delta|}).$$

# Numerical example

Let $|\{n : \alpha_n \neq 0\}| = 25$, $\{\phi_n\}$ be Haar wavelets and set

$$f(x) = \sum_{n=1}^{200} \alpha_n \phi_n(x) + \chi_{[\frac{1}{2}, \frac{9}{16}]}(x) \cos 2\pi x, \quad x \in [0, 1].$$

| $M$ | $(a)$ | $(b)$ | $(c)$ (avg. 20 trials) |
|---|---|---|---|
| 601 | 1.43e0 | 4.74e-5 | 4.73e-5 ($m = 230$) |
| 1201 | 8.5e-1 | 2.36e-5 | 2.38e-5 ($m = 460$) |

Error for (a) the partial Fourier series, (b) generalized sampling and (c) generalized sampling with compressed sensing.

# References

## Generalized sampling


- B. Adcock and A. C. Hansen, *A generalized sampling theorem for stable reconstructions in Hilbert spaces*. J. Fourier Anal. Appl. (to appear), 2011.
- B. Adcock and A. C. Hansen, *Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon*. Appl. Comput. Harmon. Anal. (to appear), 2011.
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## Generalized sampling with compressed sensing


- B. Adcock and A. C. Hansen, *Generalized sampling and infinite-dimensional compressed sensing*. Preprint, 2011.