After viewing the film The Big Short the following text was added:

This talk was given early afternoon Monday March 17, 2008.

The investment bank Bear Stearns had been in serious trouble in the week before and was given the weekend to hammer out a deal.

The first near collapse and first bailout of the 2008 financial crisis was announced due to the time shift with NY during this talk. The audience audibly reacted.

The second day of the conference took place in the Commerzbank headquarter in Frankfurt downtown, in the heart of the German financial district.
Optimization Software for Financial Mathematics

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Frankfurt MathFinance Conference
17-18 March 2008
Outline

1. The Sources
2. Classical Problems
   - LP - Linear Programming
   - QP - Quadratic Programming
   - NLP - Nonlinear Programming
3. Modern Formulations
   - CO - Conic Optimization
   - RO - Robust Optimization
4. More "Programming"
   - DP - Dynamic Programming
   - SP - Stochastic Programming
   - IP - Integer Programming
Our Sources
One book and three websites

The book

The websites
Decision Tree for Optimization Software (our share: 100%)
http://plato.asu.edu/guide.html
Benchmarks for Optimization Software (our share: 100%)
http://plato.asu.edu/bench.html
NEOS Server for Optimization (our share: 30%)
http://neos.mcs.anl.gov
The substructure in each area

Five parts

- The Problem: Mathematical Formulation
- Financial Mathematics Applications
- Commercial and "free" software
- interactive NEOS solvers
- Optimization Software Benchmarks
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**Mathematical Formulation**

**Objective**

\[ \text{min } c^T x = \min \sum_{i=1}^{N} c_i \ast x_i \]

**Variables**

\[ x_i, \quad i = 1, \ldots, N \]

**Constraints**

\[ A \ast x = b, \quad x \geq 0 \]
LP - Linear Programming

FinMath Application I

Asset/Liability Cash Flow Matching (book 3.4)

- **objective**: maximize wealth (at the end)
- **variables**: assets, liabilities (monthly)
- **constraints**: meet net cash flow
Asset Pricing and Arbitrage
Tax Clientele Effects in Bond Portfolio Management (book 4.4)

- **goal**: construct optimal tax-specific bond portfolio for a given tax bracket exploiting price differential of after tax cash flows

- **objective**: \( \text{max} \ (\text{bid-price} - \text{asking-price}) \)

- **variables**: amounts of bonds

- **constraints**: future cash flow nonnegative, bounded variables, adjustment for taxes
Commercial and "free" packages/codes

- Many commercial and free codes solve LP

**Commercial:** CPLEX, XPRESS-MP, LINDO, MOSEK, FortMP etc

**Free:** BPMPD, PCx (IPM) - CLP, QSOpt, SOPLEX (Simplex) etc

- With modeling language: All commercial, BPMPD-AMPL, PCx-AMPL, BDMP-GAMS

- Sensitivity analysis: Best in commercial codes such as CPLEX
Welcome to the NEOS server.

Our optimization solvers represent the state-of-the-art in optimization software. Optimization problems are solved automatically with minimal input from the user. Users only need a definition of the optimization problem; all additional information required by the optimization solver is determined automatically.

- User Feedback
- FAQ - NEOS Server
- Acknowledgements
- Collaborators

To submit your optimization job, first click on the **NEOS Solvers** icon to find a suitable solver.
LP solvers at NEOS

* BDMLP [GAMS Input]
* `bpmpd` [AMPL Input][LP Input][MPS Input][QPS Input]
* Clp [MPS Input]
* FortMP [MPS Input]
* MOSEK [AMPL Input][GAMS Input][MPS Input]
* OOQP [AMPL Input]
* PCx [AMPL Input][MPS Input]
* XpressMP [MOSEL Input][MPS Input]
Which benchmarks are available?

- Benchmark of commercial LP solvers (2-17-2008)
  CPLEX, MOSEK, LOQO, LIPSOL (Matlab)

- Benchmark of free LP solvers (8-7-2007)
  BPMPD, CLP, LPABO, LPAKO, QSOPT, SOPLE, GLPK

- Large Network-LP Benchmark (commercial vs free) (2-5-2008)
  CPLEX, MOSEK, CLP, QSOPT
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4 More "Programming"
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QP - Quadratic Programming

The Problem

Mathematical Formulation

**Objective**
\[ \min x^T Q x = \min \sum_{i,j=1}^{N} Q_{i,j} \ast x_i \ast x_j \] (convex)

**Variables**
\[ x_i, \quad i = 1, \ldots, N \]

**Constraints**
\[ A \ast x = b, \quad x \geq 0 \]
Portfolio Selection, Asset Allocation (book 8.1)

- $n$ number of securities
- $x_i$ share invested in security $i$
- $\mu_i$ expected return of security $i$
- $\sigma_i$ variance of security $i$
- $Q_{i,j} = \sigma_i \ast \sigma_j$
- **goal** make portfolio efficient
Portfolio Selection, Asset Allocation (book 8.1)

- either maximize return for given variance
- or minimize variance for given return $R$

Objective: $\min x^T Q x = \min \sum_{i,j=1}^N Q_{i,j} \cdot x_i \cdot x_j, \quad Q_{i,j} = \sigma_i \cdot \sigma_j$

Constraints: $e^T x = 1, \quad \mu^T x \geq R, \quad x \geq 0, \quad e^T = (1, \ldots, 1)$

- short sales allowed if $x \geq 0$ is dropped
Further applications & formulations

- include transaction costs
- maximizing the Sharpe ratio (book 8.2)

To reduce sensitivity of the QP
- use robust optimization (below)
- use Black-Litterman model (replace $\mu$ by product of distributions)
Software

Commercial and "free" packages/codes

- Several commercial and free codes solve QP
  - **Commercial**: CPLEX, XPRESS-MP, LINDO, MOSEK, LOQO, FortMP etc
  - **Free**: BPMPD, OOQP, GALAHAD, QPABO, QPS, CLP etc
- With modeling language: All commercial, BPMPD-AMPL, OOQP-AMPL,
- Sensitivity analysis: Best in commercial codes such as CPLEX
QP solvers at NEOS

* `bpmpd` [AMPL Input][LP Input][MPS Input][QPS Input]
* Clp [MPS Input]
* FortMP [MPS Input]
* MOSEK [AMPL Input][GAMS Input][MPS Input]
* OOQP [AMPL Input]
* XpressMP [MOSEL Input][MPS Input]
* All NLP solvers, two SDP solvers
Which benchmarks are available?

- Benchmark of commercial and other (QC)QP Solvers (2-5-2008)
  BPMPD, CPLEX, KNITRO, IPOPT, MOSEK, OOQP, QPB, LOQO

- AMPL-NLP Benchmark,
  IPOPT, KNITRO, LOQO, PENNLP, SNOPT (10-2-2007)
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NLP - Nonlinear Programming

The Problem

Mathematical Formulation

objective \quad \min f(x) \quad \text{(possibly not convex)}

variables \quad x_i, \quad i = 1, \ldots, N

constraints \quad g_i(x) = 0, \quad i = 1, \ldots, m

constraints \quad g_i(x) \geq 0, \quad i = m + 1, \ldots, p
Volatility Estimation (book 6.1)

- **objective**: optimize parameters in models to fit observed data (log-likelihood function)

- **constraints**: autoregressive time series, nonnegative residuals (GARCH)

- non-convex, unconstrained (recursive), nonlinearly constrained
Estimating Volatility Surface (book 6.2)

- mainly work by T. Coleman and coworkers
- Brownian motion model (for security movements)
- partial differential equations for European options
- estimate of volatility $\sigma$ of underlying security needed
- leads to nonlinear least squares problems
- dependence on $\sigma$ complicated; needs automatic differentiation
NLP - Nonlinear Programming

Software

Commercial and "free" packages/codes

- **Commercial**: KNITRO, SNOPT, LOQO, PENNLP, CONOPT, MOSEK (convex) etc

- **Free**: IPOPT, GALAHAD, HQP, SQPlab etc

- With modeling language: All commercial, IPOPT-AMPL, IPOPT-Matlab, SQPlab-Matlab
NLP solvers at NEOS

* CONOPT [GAMS Input]
* filter [AMPL Input]
* Ipopt [AMPL Input]
* KNITRO [AMPL Input][GAMS Input]
* LANCELOT [AMPL Input][SIF Input]
* LOQO [AMPL Input]
* MINOS [AMPL Input][GAMS Input]
* MOSEK [AMPL Input][GAMS Input]
* PATHNLP [GAMS Input]
* PENNON [AMPL Input]
* SNOPT [AMPL Input][FORTRAN Input][GAMS Input]
Which benchmarks are available?

- Benchmark of commercial and other (QC)QP Solvers (2-5-2008)

- AMPL-NLP Benchmark, IPOPT, KNITRO, LOQO, PENNLP, SNOPT (10-2-2007)
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CO - Conic Optimization

The Problem

Mathematical Formulation

**objective** \[ \min c^T x = \min \sum_{i=1}^{N} c_i \ast x_i \]

**constraints** \[ A \ast x = b, \quad x \text{ in cone } C \]

**SOCP cone** \[ x_1^2 \geq x_1^2 + \ldots + x_N^2, \quad x_1 \geq 0 \]

**SDP cone** \[ X \text{ symmetric positive definite, } \quad X = \text{vec}(x) \]
Tracking Errors (book 10.1)

- **goal**: track difference between return of given portfolio and benchmark XDM (market index)

- the variance-constrained Markowitz QP becomes

- **objective**: \( \min \mu^T (x - x_{XDM}) \)

- **constraints**: \( (x - x_{XDM})^T \Sigma (x - x_{XDM}) \leq TE^2 \) (SOCP)

- **constraints**: \( A \ast x = b, \quad C \ast x \geq d \)

- TE tracking error, \( \Sigma \) covariance matrix
Approximating Covariance Matrix (book 10.2)

- Not estimating Covariance Matrix
- objective \( \min \| \Sigma - \tilde{\Sigma} \|, \ \tilde{\Sigma} \text{ in the SDP cone } C \)
- constraints \( \lambda_{\min}(\tilde{\Sigma}) \geq \delta > 0 \)
Further Applications

- Recovering risk-neutral probability from options prices (book 10.3)
- Arbitrage bounds for forward start options (book 10.4)
Commercial and "free" packages/codes

- **SOCP**
  - **Commercial**: CPLEX, MOSEK
  - **Free**: SDPT3, SeDuMi, YALMIP
  - With modeling language: All commercial, CVX, YALMIP (Matlab), CVXMOD (Python)

- **SDP**
  - **Free**: SDPT3, SeDuMi, YALMIP
  - With modeling language: CVX, YALMIP (Matlab), CVXMOD (Python)
Conic solvers at NEOS

* csdp [MATLAB BINARY Input][SPARSE SDPA Input]
* DSDP [SDPA Input]
* penbmi [MATLAB Input][MATLAB BINARY Input]
* pensdp [MATLAB BINARY Input][SPARSE SDPA Input]
* sdpa [MATLAB BINARY Input][SPARSE SDPA Input]
* sdpa-c [MATLAB BINARY Input][SPARSE SDPA Input]
* sdplr [MATLAB BINARY Input][SDPLR Input][SPARSE SDPA Input]
* sdpt3 [MATLAB BINARY Input][SPARSE SDPA Input]
* sedumi [MATLAB BINARY Input][SPARSE SDPA Input]
Which benchmarks are available?

- Several SDP-codes on SDP problems with free variables (7-23-2007)
- Several SDP codes on problems from SDPLIB (8-10-2007)
- SQL problems from the 7th DIMACS Challenge (8-8-2002)
- Newer SDP/SOCP-codes on the 7th DIMACS Challenge problems (10-24-2007)
- Several SDP codes on sparse and other SDP problems (8-8-2007)
- SOCP (second-order cone programming) Benchmark (10-28-2007)
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Any Problem with Data Uncertainty

Goal

Optimize for worst case scenario varying parameters in sets

Constraint or model robustness

Solution must always be feasible ($p$ uncertain)

- Original problem: $\min f(x)$ subj to $G(x, p)$ in $K$
- Robust problem: $\min f(x)$ subj to $G(x, p)$ in $K$ for all $p$ in $U$

Objective or solution robustness

Solution must stay close to optimal ($p$ uncertain)

- Original problem: $\min f(x, p)$
- Robust problem: $\min_x \max_p f(x, p)$
RO - Robust Optimization
Examples, FinMath Application

Example I  Uncertain LP

\[
\begin{align*}
\text{min } c^T x & \quad \text{subject to } a^T x \leq b \\
a, b \text{ varying in ellipses leads to SOCP}
\end{align*}
\]

Example II  Uncertain QCLP

\[
\begin{align*}
\text{min } c^T x & \quad \text{subject to } x^T Q x + b^T x + c \leq 0 \\
Q, b, c \text{ varying in ellipses leads to SDP}
\end{align*}
\]

Application: Robust Portfolio Selection (book 20.3)

Return \( \mu \) and covariance \( \Sigma \) vary in intervals

\[
U = \{(\mu, \Sigma), \quad \mu^L \leq \mu \leq \mu^U, \quad \Sigma^L \leq \Sigma \leq \Sigma^U, \quad \Sigma \succeq 0\}
\]
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DP - Dynamic Programming

The Problem

Ingredients

- decision stages
- decision states in each stage
- transitions from state to state
- value functions per state, best possible objective
- recursive relation between value functions

The Problem

Decision Maker decides in each state -> policy or strategy

Goal: achieve best overall objective

deterministic DP if decisions define policy uniquely
Bellman’s Principle

- All local decisions have to be optimal to achieve global optimality
- forward/backward recursion

Option Pricing (book 14.1/2)

- American options (random walk, binomial lattice models)

Structuring asset-backed securities (book 15.3)

- Collateralized mortgage obligation, CMO
- CMOs make cash flow more predictable by restructuring into bonds with different maturities
few general packages

- **Commercial**: P4 (Excel add-in)
- **Free**: OpenDP, DP2PN, RDP, CompEconTB
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**SP - Stochastic Programming**

**The Problem**

**Mathematical Formulation**

- **objective**: \( \max a^T x + E(\max_y c(\omega)^T y(\omega)) \)
- \( \omega \) random event, \( E \) expected value
- **constraints**: \( Ax = b, \quad B(\omega)x + C(\omega)y(\omega) = d(\omega) \)
- **constraints**: \( x \geq 0, \quad y(\omega) \geq 0 \)

**Classification**

- This is a **two-stage SP with recourse**
- Analogously one defines multi-stage stochastic programs
SP - Stochastic Programming
FinMath Applications

Value-at-Risk, VaR, Conditional VaR

- CVaR: expected loss $\geq$ VaR (book 17.2)
- Example: Bond Portfolio Optimization (book 17.3)

Further applications

- Asset/Liability Management (book 18.1)
- Synthetic Options (book 18.2)
- Option Pricing with Transaction Costs (book 18.3)
Commercial and "free" packages/codes

**Commercial**: SLP-IOR (GAMS), SPInE, XPRESS-SP, FortSP etc

**Free**: MSLiP, SMI (COIN-OR), BNBS

Stochastic solvers at NEOS

* bnbs [SMPS Input]
* ddsip [LP Input][MPS Input] (integer variables)
* FORTSP [SMPS Input]
* MSLiP [SMPS Input]
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Mathematical Formulation

Any problem in which some variables have to attain integer values

Example I: MILP- mixed integer LP (book 12.1)

- **Combinatorial Auction**
  - $M = \{i = 1, \ldots, m\}$ items to be auctioned
  - $B_j = (S_j, P_j)$ bid, $S_j$ subset of $M$, $P_j$ price offer, $j = 1, \ldots, n$
  - objective $\max \sum_j P_j x_j$ revenue
  - constraints $\sum_j x_j \leq 1$, $i$ in $S_j$, $i = 1, \ldots, m$
  - constraints $x_j$ in $\{0, 1\}$, $1/0$ if $B_j$ wins/loses
Example II: MIQP - mixed integer QP (book 12.4)

- **Portfolio Optimization with Minimum Transaction Levels**

- Problem: investment $x_j$ too small to be executed

- $\min x^T Qx$, subj to $\mu^T x \geq R$, $Ax = b$, $Cx \geq d$

- and if $x_j > 0$ then $x_j \geq l_j$ (min transaction level)

- use standard branch&bound (B&B): find $j$ with $x_j < l_j$

- solve 2 QPs, one with $x_j = 0$ and one with $x_j \geq l_j$

- if not optimal, find next $x_j$, repeat

- Other applications: The lockbox problem (book 12.2), constructing an index fund (book 12.3)
IP - Integer Programming

Software

Codes for MILP, partly MIQP/MIQCQP

- **Commercial**: CPLEX, XPRESS-MP, LINDO, FortMP etc
- **Free**: SCIP, CBC, GLPK, MINTO, SYMPHONY, LP_SOLVE etc

Codes for MINLP

- **Commercial**: DICOPT, SBB, LINDO, BARON etc
- **Free**: BONMIN, FilmINT, MINLP, LaGO etc
Integer solvers at NEOS

* Cbc [AMPL Input][MPS Input]
* feaspump [AMPL Input][CPLEX Input][MPS Input]
* MINTO [AMPL Input]
* qsopt_ex [AMPL Input][LP Input][MPS Input]
* scip [AMPL Input][CPLEX Input][MPS Input][ZIMPL Input]
* XpressMP [GAMS Input][MOSEL Input][MPS Input]
* Bonmin [AMPL Input]
* FilMINT [AMPL Input]
* MINLP [AMPL Input]
* SBB [GAMS Input]
Which benchmarks are available?

MILP Benchmark - free codes (2-27-2007)

Feasibility Benchmark - Feaspump vs CPLEX&SCIP (2-17-2008)

MI(QC)QP Benchmark (10-28-2007)
Graphical summary of our MILP benchmark

From The SCIP webpage `scip.zib.de`:

Geometric mean of results taken from the homepage of Hans Mittelmann (2/27/2008)
Thank you!

Questions?