# Numerical Methods for Rapid Computation of PageRank

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Joint work with Chen Greif

# Outline

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# Stationary Distribution Vector of a Transition Probability Matrix

We are seeking a row vector $\pi^T$ that satisfies $\pi^T = \pi^T P$ where $P$ is a square *stochastic* matrix, with nonnegative entries between 0 and 1, and $Pe = e$, where $e$ is a vector of all-ones.

**Theorem**

*Perron(1907)-Frobenius(1912): A nonnegative irreducible matrix has a simple real eigenvalue equal to its spectral radius, whose associated eigenvector is a vector all of whose entries are nonnegative.*

What happens when $P$ is stochastic and possibly reducible?

# What Is PageRank?

**Definition**

Given a Webpage database, the PageRank of the *i*th Webpage is the *i*th element $\pi_i$ of the stationary distribution vector $\pi$ that satisfies $\pi^T P = \pi^T$, where $P$ is a matrix of *weights* of webpages that indicate their importance.

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1. $P$ is too large (size possibly in the billions) for forming any of our favorite decompositions.
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How do we modify $P$ so that there is a unique solution?

## Links determine the importance of a webpage

The fundamental idea of Brin & Page: Importance of a webpage is determined not by its contents but rather by which pages link to it. Apply the power method to a web link graph.

## Some issues with web link graphs

### Difficulties

1. The existence of dangling nodes (correspond to an all-zero row in the matrix): could have very important pages that have no outlinks. (e.g. the U.S. constitution!)
2. Periodicity: a cyclic path in the Webgraph. (e.g. You point only to your mom's webpage and she points only to yours.)

Simple example:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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## Solution

*Set $M(c) = cP + (1 - c)E$, where $E$ is a positive rank-1 matrix.*

## The matrix $M(c)$

- We have $M(c) > 0$ which yields a unique solution. But what is the significance of the stationary probability vector?
- $M(c)$ is a Markov chain with positive entries, and

$$M(c)z(c) = z(c).$$

Therefore for $c < 1$, $z(c)$ is unique (under proper scaling).

# Simple example (Glynn and G.)

For the identity matrix, $P = I$, no unique stationary probability distribution, but for $M(c) = cI + (1-c)ee^T/n$ we are converging to

$$z(c) = \frac{1}{n}e.$$

## The significance of the parameter $c$

- $c$ is the probability that a surfer will follow an outlink (as opposed to jump randomly to another Webpage).
- $c = 0.85$ was the choice in the Brin & Page model.
- Like regularization: small value leads to a more stable computation, but further away from true solution.

# Brin & Page's Strategy: Apply Power Method

For Google, it all boiled down originally to solving the eigenvalue problem

$$x = Mx$$

using the power method

$$x^{(k+1)} = Mx^{(k)}.$$

# Discussion

Let $Mz_i = \lambda_i z_i$. For $|\lambda_i| \neq |\lambda_j|$ we have

$$x^{(0)} = \sum \alpha_i z_i,$$

and

$$x^{(k)} = \sum \alpha_i \lambda_i^k z_i,$$

with $\|x^{(k)}\|_1 = 1$ and $x \geq 0$.
After normalization, for $\lambda_1 = 1$ we have

$$x^{(k)} = z_1 + \sum_{j=2}^{n} \beta_j \lambda_j^k z_j.$$

# The Eigenvalues of the PageRank Matrix

**Theorem**

*(Elegant proof due to Eldén)*
*Let $P$ be a column-stochastic matrix with eigenvalues $\{1, \lambda_2, \lambda_3, \dots, \lambda_n\}$. Then the eigenvalues of $M(c) = cP + (1-c)ve^T$, where $0 < c < 1$ and $v$ is a nonnegative vector with $e^T v = 1$, are*

$$\{1, c\lambda_2, c\lambda_3, \dots, c\lambda_n\}.$$

This implies

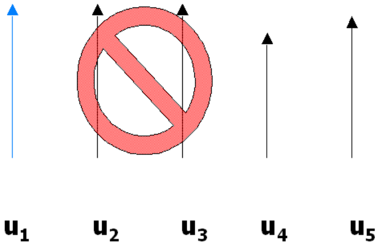
$$\frac{|\lambda_j|}{|\lambda_1|} \leq c.$$

# Outline

# Quadratic Extrapolation (Kamvar, Haveliwala, Manning, G.)

Slowly convergent series can be replaced by series that converge to the same limit at a much faster rate.

**Idea:** Estimate components of current iterate in the directions of second and third eigenvectors, and eliminate them.



# Quadratic Extrapolation

Suppose $M$ has three distinct eigenvalues.
The minimal polynomial is given by

$$P_M(\lambda) = \gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 + \gamma_3\lambda^3.$$

By the Cayley-Hamilton theorem, $P_M(M) = 0$. Hence for any vector $z$,

$$P_M(M)z = (\gamma_0 + \gamma_1 M + \gamma_2 M^2 + \gamma_3 M^3)z = 0.$$

## Quadratic Extrapolation (cont.)

Set $z = x^{(k-3)}$ and use the fact that $x^{(k-2)} = Mx^{(k-3)}$ and so on. Thus,

$$(x^{(k-2)} - x^{(k-3)})\gamma_1 + (x^{(k-1)} - x^{(k-3)})\gamma_2 + (x^{(k)} - x^{(k-3)})\gamma_3 = 0.$$

Defining

$$y^{(k-j)} = x^{(k-j)} - x^{(k-3)}, \quad j = 1, 2, 3,$$

and setting $\gamma_3 = 1$ (to avoid getting a trivial solution $\gamma = \mathbf{0}$), get

$$(y^{(k-2)} \ y^{(k-1)})[\gamma_1 \ \gamma_2]^T = -y^{(k)}.$$

Now, since $M$ has more than three eigenvalues, solve a least squares problem.

# The dynamic nature of the web

This problem involves a matrix which is changing over time.

- States increase and decrease, i.e. new websites are introduced and old websites die.
- Websites are continually changing. $M$ is a function of time and so is its dimension.

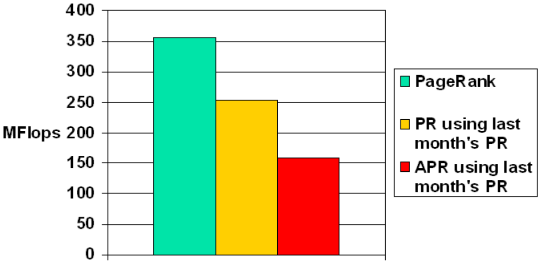
# Adaptive Computation (joint with Kamvar and Haveliwala)

Most pages converge rapidly. Basic idea: when the PageRank of a page has converged, stop recomputing it.

$$x_N^{(k+1)} = M_N x^{(k)} \; ;$$
$$x_C^{(k+1)} = x_C^{(k)}.$$

- Use the previous vector as a start vector.
- Nice speedup, but not great. Why? The old pages converge quickly, but the new pages still take long to converge. Web constantly changes! Addition, deletion, change of existing pages...
- But, if you use Adaptive PageRank, you save the computation of the old pages.

# Example: Stanford-Berkeley, $n \approx 700000$



# Other Effective Approaches

- Aggregation/Disaggregation. (Stewart, Langville & Meyer, .....)
- Approaches related to permutations of the Google matrix. (Del Corso et. al., Kamvar et. al.)
- Linear system formulation. (Arasu et. al.)

and more...
Survey paper:
*A survey of eigenvector methods of Web information retrieval*
by Amy Langville and Carl Meyer.
Stability and convergence analysis: Ipsen & Kirkland.

# Outline

# Using the Arnoldi method for PageRank (joint with Chen Greif)

Arnoldi method:

The Arnoldi method is generally used for generating a small upper Hessenberg that approximates some of the eigenvalues of the original matrix. When $Q$ is orthogonal,

$$Q^T MQ(Q^T x) = \lambda(Q^T x).$$

1. Find $H = Q^T MQ$ upper Hessenberg, then perform the computations for $H$ instead of $M$.
2. $M$ is $n$-by-$n$ and is huge, but we terminate the process after $k$ steps. Resulting $H$ is $(k+1)$-by-$k$.

# Computational Cost

1. Main cost: One matrix-vector product (with original large matrix) per iteration.
2. Inner products and norm computations.
3. Power method cheaper but not by much if matrix-vector products dominate.

# An Arnoldi/SVD algorithm for computing PageRank

Similar to computing *refined Ritz vectors* (Jia, Stewart), but pretend largest eigenvalue stays 1 in smaller space, i.e. we do not compute any Ritz values.

Set initial guess $q$ and $k$, the Arnoldi steps number
Repeat
.....$[Q, H] = Arnoldi(A, q, k)$
.....Compute $H - [I; 0] = U\Sigma V^T$
.....Set $v = V(:, k)$
.....Set $q = Qv$
Until $\sigma_{\min}(H - [I; 0]) < \varepsilon$

# Advantages

- Orthogonalization achieves effective separation of eigenvectors.
- Take advantage of knowing the largest eigenvalue.
- Largest Ritz value could be complex, but if we set the shift to 1 then no risk of complex arithmetic.
- Smallest singular value converges smoothly to zero (more smoothly than largest Ritz value converges to 1).
- Stopping criterion with no computational overhead:

$$\|Aq - q\|_2 = \sigma_{\min}(H - [I; 0]).$$

## Disadvantages

- More complicated to implement.
- A single iteration is more expensive than a power iteration; must converge within fewer iterations.

# Sensitivity of the PageRank Vector

$$M(c) = cP + (1-c)ev^T ; e = [1, \ldots, 1]^T \ , \quad v = \frac{e}{n}.$$

$$M(c)x(c) = x(c);$$

$$M'x + Mx' = x';$$

$$M' = P - ev^T = \frac{1}{c}(M - ev^T);$$

$$(I - M)x' = M'x = \frac{1}{c}(x - v).$$

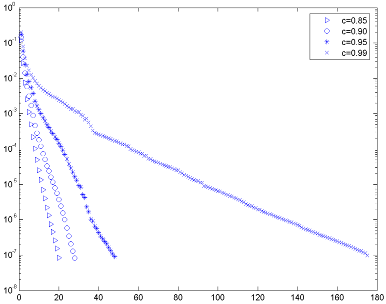
Get the exact same matrix, $I - M$: singular *consistent* linear system. Goal: identify 'sensitive' vs. 'insensitive' components.
Difficulty: How do we compute it?

# Web Matrices

| name | size | nz | avg nz per row |
|---|---|---|---|
| sg | 3,685 | 32,445 | 8.8 |
| bs | 19,566 | 133,535 | 6.8 |
| Stanford | 281,903 | 2,312,497 | 8.2 |
| Stanford-Berkeley | 683,446 | 7,583,376 | 11.1 |
| Wikipedia | 1,104,857 | 18,265,794 | 16.5 |
| edu | 2,024,716 | 14,056,641 | 6.9 |

Thanks for David Gleich and Yahoo! Inc.

# Effect of the damping factor $c$




Typical behavior for the test matrices: difference in convergence rate is significant.

# Numerical example

| $c$ | Power | $k = 4$ | $k = 8$ | $k = 16$ |
|---|---|---|---|---|
| 0.85 | 77 | 76 | 64 | 64 |
| 0.90 | 117 | 112 | 96 | 80 |
| 0.95 | 236 | 192 | 136 | 114 |
| 0.99 | 1165 | 700 | 504 | 352 |

Matrix-vector products for various values of the damping factor $c$, for the $281903 \times 281903$ Stanford matrix. The stopping criterion was $\|x^{(k)} - Ax^{(k)}\|_1 < 10^{-7}$.

## Ordering is a function of $c$ (a few rankings in Wikipedia)

| Entry | $c = 0.85$ | $c = 0.90$ | $c = 0.95$ | $c = 0.99$ |
|---|---|---|---|---|
| United States | 1 | 1 | 1 | 1 |
| Race (U.S. Census) | 2 | 2 | 4 | 20 |
| United Kingdom | 3 | 3 | 2 | 2 |
| France | 4 | 4 | 5 | 7 |
| 2005 | 5 | 5 | 11 | 10 |
| 2004 | 6 | 6 | 12 | 13 |
| 2000 | 7 | 15 | 20 | 29 |
| Canada | 8 | 10 | 17 | 17 |
| Category: culture | 12 | 9 | 8 | 6 |
| Category: politics | 13 | 7 | 6 | 5 |
| Category: wikiportals | 18 | 8 | 3 | 3 |
| Italy | 28 | 27 | 31 | 40 |
| Sweden | 80 | 92 | 94 | 100 |

# Observations

- Top ranked entry stays on top throughout.
- Countries generally lose ground as $c$ goes up; categories make gains.
- Second ranked entry for $c = 0.85$ [Race (U.S. census)] is ranked 20th for $c = 0.99$.
- On the other hand 18th ranked entry for $c = 0.85$ is ranked third for $c = 0.99$. [`Wikiportals` are pages functioning as a portal for a particular subject area.]
- Other entries also show change in ranking as a function of the damping factor.

# Changes at the top as a function of $c$

| $c$ | top 5 | top 10 |
|---|---|---|
| 0.85 | 5 | 10 |
| 0.90 | 5 | 7 |
| 0.95 | 3 | 4 |
| 0.99 | 2 | 3 |

Match of webpages in the top rankings. The top 5 (first column) and top 10 (second column) pages for $c = 0.85$ were taken, and in the table the numbers indicate how many of them appear in the top 5 and 10 for other values of $c$.

# Summary

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- Decomposition-free methods are necessary.
- Techniques for convergence acceleration prove effective.
- For $c = 0.85$ power method seems good enough, but not for higher values of $c$.
- Arnoldi approach seems a natural way to go and proves effective.

## Challenges

- How to determine the reliability of PageRank by means of sensitivity.
- Efficient methods for a large value of $c$.