Verification of Second-Order Sufficient Optimality Conditions for Semilinear Elliptic and Parabolic Control Problems

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Abstract: We study optimal control problems for semilinear parabolic equations subject to control constraints and for semilinear elliptic equations subject to control and state constraints. We quote known second-order sufficient optimality conditions (SSC) from the literature. Both problem classes, the parabolic one with boundary control and the elliptic one with boundary or distributed control, are discretized by a finite difference method. The discrete SSC are stated and numerically verified in all cases providing evidence of optimality where only necessary conditions had been studied before.

Key words: Elliptic and parabolic control problems, control and state constraints, second-order sufficient conditions, discretizations, optimization methods

1 Introduction

A wealth of literature exists on theoretical and computational aspects of control problems for ordinary differential equations. This includes in particular results on necessary and sufficient optimality conditions. Exemplarily, we mention the fundamental works [9, 11] in which it was shown that a *two-norm discrepancy* cannot be avoided in the continuous case. Research on control problems governed by partial differential equations has started more recently as did the investigation of optimality conditions for these problems. First, nonlinear parabolic control problems were considered, for example, in [7], then elliptic problems [3]. Some of the most recent work is [4, 15] in which such problems with control and state constraints are treated. The establishment of second-order sufficient optimality conditions (SSC) requires a highly technical machinery including the use of a third norm as was done before in [5, 10].

The verification of the, partly restrictive, continuous optimality condition is possible only in very rare circumstances, such as an analytically known solution for which all conditions can be evaluated analytically; for a parabolic example, see [1]. For the numerical solution of such control problems there are two different approaches, a direct discretization of the entire problem leading to a large finite-dimensional constrained optimization problem. This approach has also been called "all-at-once" method, see, for example, [18], "one-shot" method, etc. An iterative method can be applied in function space resulting in a series of infinite-dimensional linear-quadratic control problems which then still need to be discretized. The literature on the latter is quite extensive while the former has more recently been considered for the solution of real-life problems, when progress had been made on solution techniques for very large constrained nonlinear programming (NLP) problems. Exemplarily, we cite the special issue [17].

In [12]–[14] we have considered general semilinear elliptic control problems with control and state constraints. The first-order necessary conditions were formally derived in the continuous case. These were compared to the finite-dimensional conditions for a direct finite-difference discretization of the control problem and the latter was solved by applying state-of-the-art interior point and SQP methods.

The direct discretization will lead to huge NLP's for three-dimensional elliptic or two and three-dimensional parabolic problems. If the discretization is consistent, preferably of higher order in the discretization parameter, the numerical verification of the SSC for these problems, even for relatively coarse discretizations, may be expected to yield clues on the optimality of the approximated continuous solution. It is in this spirit that below a class of both parabolic and elliptic control problems in one respectively two space dimensions is considered. The problems are discretized and the SSC are verified. Since the parabolic problem is that from [1, 7] and the elliptic problems are the ones from [12]–[14] with the exception of the problems for which singular or bang-bang controls were obtained, our results are supplementing those in these papers.

In the following section both the parabolic and elliptic control problems are stated. Some known SSC results are quoted in the third section. In section 4 the way in which the discrete SSC are verified is described followed by results for cases from the above sources in section 5. Concluding remarks are made in the last section.

2 Parabolic and Elliptic Control Problems

In this section we describe the continuous control problems *exemplarily* considered in the following. The first is a one-dimensional parabolic boundary control problem formulated such that it includes problems from [1, 7]

$$
\begin{aligned}
& \text{minimize} \\
f(y,u) \quad = \quad & \frac{1}{2}\int_0^l (y(x,T)-y_T(x))^2dx + \frac{\alpha}{2}\int_0^T u(t)^2dt \\
& + \int_0^T (a_y(t)y(l,t)+a_u(t)u(t))dt, \quad \alpha > 0
\end{aligned}
$$

subject to (P)

$$
\begin{aligned}
& y_t - y_{xx} = 0 \quad \text{in } (0,l)\times(0,T) \\
& y(x,0) = a(x) \quad \text{in } (0,l) \\
& y_x(0,t) = 0 \quad \text{in } (0,T) \\
y_x(l,t) \quad + \quad & \beta y(l,t) = b(t) + u(t) - \varphi(y(l,t)) \quad \text{in } (0,T) \\
& \alpha_1 \le u(t) \le \alpha_2 \quad \text{in } (0,T).
\end{aligned}
$$

The notation above is that of [1] and the corresponding data will be defined in section 5 below. The problem does, however, include also one considered in [7], namely by choosing $b(t)=0$, $\varphi(y(l,t)) = y^2(l,t), a_y(t)=0, a_u(t)=0$.

We define the following discretization of problem (P).

$$
\begin{aligned}
& \text{minimize} \quad f_h(y_h,u_h) = \frac{dx}{4}((y_{0,m}-y_T(x_0))^2 \\
& +2\sum_{j=1}^{n-1}(y_{j,m}-y_T(x_j))^2 + (y_{n,m}-y_T(x_n))^2) + \frac{\alpha dt}{4}\Big(2\sum_{j=1}^{m-1}u_j^2 + u_m^2\Big) \\
& +dt\Big(\sum_{j=1}^{m-1}(a_y(t_j)y_{n,j}+a_u(t_j)u_j) + \frac{1}{2}a_y(T)y_{n,m} + a_u(T)u_m\Big)
\end{aligned}
$$

subject to (P_h)

$$
\begin{aligned}
\frac{y_{j,i+1}-y_{j,i}}{dt} &= \frac{1}{2}(y_{j-1,i}-2y_{j,i}+y_{j+1,i} \\
& +y_{j-1,i+1}-2y_{j,i+1}+y_{j+1,i+1})/dx^2
\end{aligned}
$$

$$
\begin{aligned}
& \quad i = 0, \ldots, m-1, \quad j = 1, \ldots, n-1 \\
& y_{j,0} = a(x_j), \quad j = 0, \ldots, n \\
& y_{2,i} - 4y_{1,i} + 3y_{0,i} = 0, \quad i = 1, \ldots, m \\
& (y_{n-2,i} - 4y_{n-1,i} + 3y_{n,i})/(2dx) + y_{n,i} \\
& = u_i + b(t_i) - \varphi(y_{n,i}), \quad i = 1, \ldots, m \\
& \alpha_1 \le u_i \le \alpha_2, \quad i = 1, \ldots, m.
\end{aligned}
$$

Here $x_j = jdx$, $dx = 1/n$, $t_j = jdt$, $dt = T/m$.

For the problem (P) above and specific data an analytic solution is given in [1] and this also permits the authors to verify the necessary and sufficient optimality conditions they had stated and proved.

For the elliptic control problems we consider the class defined in [14]. It includes boundary and distributed controls which are addressed separately in [12], respectively [13] as well as Dirichlet, Neumann, and mixed boundary conditions. In the case of boundary control the underlying continuous problem is

$$
\begin{aligned}
& \text{minimize} \\
& F(y,u) = \int_{\Omega_0} f(x,y(x))dx + \int_{\Gamma_1} g(x,y(x),u(x))dx \\
& \qquad + \int_{\Gamma_2} k(x,u(x))dx
\end{aligned}
$$

subject to (EB)

$$
\begin{aligned}
& -\Delta y(x) + d(x,y(x)) = 0, \quad \text{for } x \in \Omega, \\
& \partial_\nu y(x) = b(x,y(x),u(x)), \quad \text{for } x \in \Gamma_1, \\
& y(x) = a(x,u(x)), \quad \text{for } x \in \Gamma_2,
\end{aligned}
$$

and

$$
\begin{aligned}
& C(x,u(x)) \le 0, \quad \text{for } x \in \Gamma, \\
& S(x,y(x)) \le 0, \quad \text{for } x \in \Omega.
\end{aligned}
$$

Here Ω is a bounded, plane domain with piecewise smooth boundary Γ. $\Omega_0 \subset \Omega$ is equal to Ω unless noted otherwise. ∂_ν denotes the derivative in the direction of the outward unit normal ν on Γ and the boundary is partitioned as $\Gamma = \Gamma_1 \cup \Gamma_2$ with disjoint sets Γ_1, Γ_2 consisting of finitely many connected components. For the general formulation given above in [13, 14] necessary optimality conditions are stated, a discretization is described in full detail and the corresponding optimality conditions are related carefully to those for the continuous problem. With concrete applications in mind then, however, the following data of the problem are specialized, the objective function

$$F(y,u)=\frac{1}{2}\int_{\Omega_0}(y(x)-y_d(x))^2dx+\frac{\alpha}{2}\int_{\Gamma}(u(x)-u_d(x))^2dx \tag{2.1}$$

with $\alpha \geq 0$ and the inequality constraints

$$y(x)\leq\varphi(x),\quad x \text{ in } \Omega,\quad u_1(x)\leq u(x)\leq u_2(x)\quad \text{on } \Gamma. \tag{2.2}$$

This last set of conditions is also considered as a special case for the *distributed* control problem in [13, 14], the data of which are

$$\begin{aligned}&\text{minimize}\\ &\qquad F(y,u)=\int_{\Omega}f(x,y(x)),u(x))dx+\int_{\Gamma_1}g(x,y(x))dx\\ &\text{subject to}\\ &\qquad -\Delta y(x)+d(x,y(x),u(x))=0,\quad x\in\Omega,\\ &\qquad \partial_\nu y(x)=b(x,y(x)),\quad x\in\Gamma_1,\\ &\qquad y(x)=y_2(x),\quad x\in\Gamma_2,\end{aligned} \tag{ED}$$

and the bounds as in (2.2).

As a special objective function

$$F(y,u)=\frac{1}{2}\int_{\Omega}(y(x)-y_d(x))^2dx+\frac{\alpha}{2}\int_{\Omega}(u(x)-u_d(x))^2dx \tag{2.3}$$

is considered.

3 Known SSC Results for the Continuous Problems

No attempt will be made to quote all relevant results on optimality conditions. In the case of problem (P) the following set of SSC is stated in [1].

First, the Lagrange function

$$\begin{aligned} L(y,u,p) &= F(y,u) - \int_0^T \int_0^l (y_t - y_{xx}) p(x,t)\,dx\,dt \\ &\quad + \int_0^T y_x(0,t) p(0,t)\,dt \\ &\quad - \int_0^T (y_x(l,t) + y(l,t)) p(l,t)\,dt \\ &\quad + \int_0^T (b(t) + u(t) - \varphi(y(l,t))) p(l,t)\,dt \end{aligned}$$

is defined with the Lagrange multiplier function p. Then the second derivative of L with respect to (y,u) is called L'' and is evaluated at a point $(\bar{y}, \bar{u}, \bar{p})$ satisfying the first order optimality conditions. The key requirement is the inequality

$$L''(\bar{y}, \bar{u}, \bar{p})(y,u)^2 \geq \|y(\cdot, T)\|^2_{L^2(0,l)} + \|u\|^2_{L^2(0,T)} \tag{3.1}$$

which has to hold for all (y,u) which satisfy the linearized (at $(\bar{y}, \bar{u})$) constraints. In the example considered in [1] and below, the inequality even holds for all (y,u). Locally, then, $(\bar{y}, \bar{u})$ is a minimizer of problem (P).

SSC have not been stated exactly for the two elliptic control problems (EB), (ED) in the previous section, but a series of papers address special cases. In [3], for example, a good overview of the literature is given and the boundary control problem covered in much technical detail is nearly identical to (EB) except for

$$d(x, y(x)) = y(x), \quad \Gamma_1 = \Gamma.$$

On the other hand, a problem of type (ED) but with

$$d(x,y,u) = u - \varphi(y), \quad \Gamma_2 = \Gamma, \quad y_2 \equiv 0$$

and the tracking type objective function (2.3) is extensively analyzed in [2]. Finally, the state-constrained case is addressed in [4] for (EB) and in [15] for (P).

4 Second Order Sufficient Conditions for the Discretized Problems

All three control problems (P), (EB), (ED) defined in the previous sections lead after suitable discretization to nonlinear finite-dimensional optimization problems of the form

$$\min F^h(z) \quad \text{subject to} \quad G^h(z) = 0, \quad H(z) \leq 0 \tag{4.1}$$

where z comprises the discretized control and state variables.

A discretization (P_h) of (P) is given in section 2 while the elliptic problems are assumed to be discretized as described in detail in [12]–[14]. $G^h(z)$ symbolizes the state equation and boundary conditions while $H(z)$ denotes both pointwise control and state constraints, the only constraints of inequality type prescribed above. Thus, alternatively, it can be written as

$$z_l \leq z \leq z_u \tag{4.2}$$

where components of $z_l(z_u)$ are taken as $-\infty(\infty)$ if no constraint is imposed on the component.

We state the well-known SSC for (4.1), assuming $z \in \mathbf{R}^n$, $G^h : \mathbf{R}^n \to \mathbf{R}^m$, $m < n$. Let z^* be an admissible point satisfying the first-order necessary optimality conditions with associated Lagrange multipliers μ^* and λ^*. Let further

$$N(z^*) = (\nabla G^h(z^*), \nabla H_a(z^*))$$

be a column-regular $n \times (m+p)$ matrix where $m+p < n$ and $\nabla H_a(z^*)$ denotes the gradients of the p active inequality constraints. Let finally $N = QR$ be a QR decomposition and $Q = (Q_1, Q_2)$ a splitting into the first $m+p$ and the remaining columns.

The point z^* is a strict local minimizer if a $\gamma > 0$ exists such that, see, for example, [19]

$$\lambda_{\min}(L_2(z^*)) = \gamma > 0. \tag{4.3}$$

Here $L_2(z^*)$ is the projected Hessian of the Lagrangian

$$L_2(z^*) = Q_2^T(\nabla^2 F^h(z^*) - \mu^{*T} \nabla^2 G^h(z^*))Q_2.$$

Next, we will detail how condition (4.3) will be checked for the discrete versions of both the parabolic and the elliptic control problems. As was already done in [12]–[14] the control problems are written in the form of AMPL [6] scripts. This way, a number of nonlinear optimization codes can be utilized for their solution. It had been an observation in our previous work that from the currently available codes only LOQO [20] is able to solve all the problems effectively and for sufficiently fine discretizations. The following is independent of the solver used.

After computing a solution an AMPL *stub* (or $*.nl$) file is written as well as a file with the computed Lagrange multipliers. This allows to check the SSC (4.3) with the help of a Fortran, alternatively, a C or Matlab, program.

This program reads the files and verifies first the necessary first-order optimality conditions, the column regularity of $N(z^*)$ and the strict complementarity. For this, it utilizes routines provided by AMPL which permit evaluation of the objective and constraint gradients. Next, it computes the QR decomposition of $N(z^*)$ with the help of LAPACK routine DGEQRF. This use of dense numerical linear algebra limits the problem size and will in future work be replaced by an approach exploiting sparsity. AMPL also provides a routine to multiply the Hessian of the Lagrangian times a vector. This is called with the columns of Q_2 and $L_2(z^*)$ can be formed. Its eigenvalues are computed with LAPACK routine DSYEV and the smallest eigenvalue $\gamma = \gamma_h$ is determined.

5 Verification of SSC Conditions

In this section numerical results will be reported for the application of the method outlined in the previous section to two parabolic control problems from [1, 7] and a total of ten elliptic control problems from [12]–[14]. The first parabolic problem is particularly interesting because for it in [1] an analytical exact solution is given and the continuous SSC conditions are verified. For the sake of completeness, the specification will be given for each problem. The discretizations used are those defined in section 2 for the parabolic and in [12]–[14] for the elliptic problems.

Problem (P) from section 2 is solved with the following data

$$\begin{aligned}
&l = \pi/4, \quad T = 1, \quad \alpha = \frac{\sqrt{2}}{2}(e^{2/3} - e^{1/3}) \\
&y_T(x) = (e + e^{-1}) \cos x, \quad \alpha_1 = 0, \quad \alpha_2 = 1 \\
&a(x) = \cos x, \quad a_y(t) = -e^{-2t}, \quad a_u(t) = \frac{\sqrt{2}}{2} e^{1/3} \\
&b(t) = \frac{1}{4} e^{-4t} - \min\left(1, \max\left(0, \frac{e^t - e^{1/3}}{e^{2/3} - e^{1/3}}\right)\right)
\end{aligned} \tag{5.1}$$

$1/h$	u-error	y-error
50	2.062e-4	4.460e-5
100	1.438e-5	1.209e-5
200	1.884e-5	3.171e-6
350	6.289e-6	1.016e-6

Table 1: Solution errors for problem 5.1

$1/h$	γ_h	γ_h/h^2
50	6.379e-4	1.595
59	4.781e-4	1.664
70	3.356e-4	1.644
80	2.577e-4	1.649
89	2.135e-4	1.691

Table 2: Minimal eigenvalue for problem 5.1

$$\varphi(y) = y|y|^3, \quad \beta = 1.$$

As is shown in [1] a local optimum for this problem is the pair $(\bar{y}, \bar{u})$,

$$\begin{aligned} \bar{y}(x,t) &= e^{-t} \cos x \\ \bar{u}(t) &= \min\left(1, \max\left(0, \frac{e^t - e^{1/3}}{e^{2/3} - e^{1/3}}\right)\right). \end{aligned}$$

The discretization (P_h) of section 2 was coded in AMPL and solved with LOQO. Afterwards, the SSC were checked as described in the previous section resulting in the minimal eigenvalue γ_h, cf. (4.3).

In Table 1 the maximum errors over the grid points are listed for the computed control and state functions while in Table 2 the smallest eigenvalue is listed as well as the value scaled by h^{-2}. The errors exhibit quadratic convergence while the scaled eigenvalue stays nearly constant with a slightly increasing tendency. These facts justify the application of the technique to other control problems for which no exact solution is known and it also shows which scaling of γ_h is appropriate for problem (P_h).

As a second parabolic case one from [7] was chosen. The data for (P) are

$$l = 1, \quad T = 1.58, \quad \alpha = .001$$

$$
\begin{array}{ll}
 & y_T(x) = .5(1-x^2), \quad \alpha_1 = -1, \quad \alpha_2 = 1 \\
 & a(x) = 0, \quad a_y(t) = 0, \quad a_u(t) = 0, \\
\text{(I)} & b(t) = 0, \quad \varphi(y) = 0, \quad \beta = 1 \\
\text{(II)} & b(t) = 0, \quad \varphi(y) = y^2, \quad \beta = 0
\end{array}
\tag{5.2}
$$

The case (I) leads to a linear-quadratic control problem and had already been considered in [16]. For both cases just the minimal eigenvalue can be listed without and with the same scaling as in the previous example.

y_h	γ_h	γ_h/h^2
60	5.604e-6	2.02e-2
70	4.215e-6	2.06e-2
80	3.249e-6	2.08e-2
90	2.607e-6	2.11e-2

Table 3: Minimal eigenvalue for 5.2-I

$1/h$	γ_h	γ_h/h^2
60	2.498e-6	8.99e-3
70	1.866e-6	9.14e-3
80	1.447e-6	9.26e-3
90	1.155e-6	9.36e-3

Table 4: Minimal eigenvalue for 5.2-II

Next, we present the data for the elliptic boundary control problems from [12, 14] and the eigenvalues obtained. The domain is the unit square in all cases. The problem (EB) together with (2.1) and (2.2) is considered and $u_d = 0$ in all cases.

(EB-1)

$$
\begin{array}{l}
d(x,y) = -20, \quad a(x,u) = u, \quad \Gamma_2 = \emptyset, \\
y_d(x) = 3 + 5x_1(x_1-1)x_2(x_2-1), \\
\alpha = .01, \quad \psi(x) = 3.5, \quad u_1 = 0, \quad u_2 = 10
\end{array}
$$

(EB-2) as above, but $\varphi(x) = 3.2$, $u_1 = 1.6$, $u_2 = 3.2$.

$1/h$	γ_h	γ_h/h
60	3.333e-4	2.0e-2
70	2.857e-4	2.0e-2
80	2.500e-4	2.0e-2
90	2.222e-4	2.0e-2

Table 5: Minimal eigenvalue for (EB-1)

$1/h$	γ_h	γ_h/h
60	3.333e-4	2.0e-2
70	2.857e-4	2.0e-2
80	2.521e-4	2.0e-2
90	2.222e-4	2.0e-2

Table 6: Minimal eigenvalue for (EB-2)

(EB-3)

$$d(x,y) = 0, \quad b(x,y,u) = u - y^2, \quad \Gamma_1 = \emptyset,$$
$$y_d(x) = 2 - 2(x_1(x_1-1) + x_2(x_2-1)),$$
$$\alpha = .01, \quad \psi(x) = 2.071, \quad u_1 = 3.7, \quad u_2 = 4.5$$

$1/h$	γ_h	γ_h/h
60	1.534e-4	9.20e-3
70	1.300e-4	9.10e-3
80	1.135e-4	9.08e-3
90	1.008e-4	9.07e-3

Table 7: Minimal eigenvalue for (EB-3)

(EB-4)

$$d(x,y) = y - y^3, \quad b(x,y,u) = u, \quad \Gamma_1 = \emptyset,$$
$$y_d(x) = 2 - 2(x_1(x_1-1) + x_2(x_2-1)),$$
$$\alpha = .01, \quad \varphi(x) = 2.7, \quad u_1 = 1.8, \quad u_2 = 2.5$$

$1/h$	γ_h	γ_h/h
60	1.654e-4	9.92e-3
70	1.412e-4	9.88e-3
80	1.232e-4	9.86e-3
90	1.094e-4	9.85e-3

Table 8: Minimal eigenvalue for (EB-4)

(EB-5)

$$
\begin{aligned}
&d(x,y) \equiv 0; \quad g(x,y,u) \equiv 0; \quad b(x,y,u) = 0, \quad \text{on} \quad x_2 = 0,\\
&b(x,y,u) = y - 5 \quad \text{on} \quad x_1 = 0, \quad x_1 = 1,\\
&a(x,u) = u \quad \text{on} \quad \{x_2 = 1\} = \Gamma_2,\\
&\psi(x) = 3.15 \quad \text{in} \quad \Omega_0, \quad \psi(x) = 10 \quad \text{in} \quad \Omega\backslash\Omega_0,\\
&u_1 = 0, \quad u_2 = 10 \quad \text{on} \quad \Gamma_2\\
&y_d \equiv 1, \quad \Omega_0 = [.25, .75]^2, \quad \alpha = .005
\end{aligned}
$$

$1/h$	γ_h	γ_h/h^2
40	1.412e-4	2.26e-1
56	7.530e-5	2.36e-1
72	4.618e-5	2.39e-1
88	3.026e-5	2.34e-1

Table 9: Minimal eigenvalue for (EB-5)

In all cases except (EB-5) which is different from the others in key elements, the scaling of γ_h by h^{-1} was appropriate. The following five elliptic distributed control examples were considered in [13, 14].

(ED-1)

$$
\begin{aligned}
&f(x,y,u) = u^2 - .8uy, \quad g(x,y) = 0,\\
&d(x,y,u) = -y(a(x) - u - y), \quad b(x,y) = 0,\\
&\Gamma = \Gamma_1, \quad a(x) = 7 + 4\sin(2\pi x_1 x_2)\\
&\psi(x) = 7.1, \quad u_1 = 1.7, \quad u_2 = 2
\end{aligned}
$$

$1/h$	γ_h	γ_h/h^2
40	1.250e-3	2.0e0
50	8.002e-4	2.0e0
60	5.557e-4	2.0e0

Table 10: Minimal eigenvalue for (ED-1)

(ED-2)

$$d(x,y,u) = -y + y^3 - u,$$
$$y_2(x) = 0, \quad \Gamma = \Gamma_2, \quad \alpha = .001,$$
$$y_d(x) = 1 + 2(x_1(x_1 - 1) + x_2(x_2 - 1)))$$
$$\psi(x) = 1.85, \quad u_1 = 1.5, \quad u_2 = 4.5$$

$1/h$	γ_h	γ_h/h^2
40	6.028e-6	9.64e-3
50	3.983e-6	9.96e-3
60	2.719e-6	9.79e-3

Table 11: Minimal eigenvalue for (ED-2)

(ED-3)

$$d(x,y,u) = \exp(y) - u,$$
$$y_2(x) = 0, \quad \Gamma = \Gamma_2, \quad \alpha = .001$$
$$y_d(x) = \sin(2\pi x_1)\sin(2\pi x_2)$$
$$\psi(x) = .11, \quad u_1 = -5, \quad u_2 = 5$$

(ED-4)

$$d(x,y,u) = \exp(y) - u, \quad b(x,) = -y, \quad \Gamma = \Gamma_1, \quad \alpha = .001$$
$$y_d(x) = \sin(2\pi x_1)\sin(2\pi x_2), \quad \psi(x) = .371, \quad u_1 = -8, \quad u_2 = 9$$

(ED-5) as (ED-1) except

$$b(x,y) = -\beta y, \quad \beta = 1 \text{ on } x_1 = 0, \quad x_2 = 0, \quad \beta = 0 \quad \text{otherwise}$$
$$\psi(x) = 6.09, \quad u_1 = 1.4, \quad u_2 = 1.6$$

$1/h$	γ_h	γ_h/h^2
40	6.250e-7	1.0e-3
50	4.000e-7	1.0e-3
60	2.777e-7	1.0e-3

Table 12: Minimal eigenvalue for (ED-3)

$1/h$	γ_h	γ_h/h^2
40	6.250e-7	1.0e-3
50	4.000e-7	1.0e-3
60	2.777e-7	1.0e-3

Table 13: Minimal eigenvalue for (ED-4)

6 Conclusion

In this work the second-order sufficient optimality conditions (SSC) were verified numerically for a number of different optimal control problems taken from five recent papers. For a parabolic problem with analytically known solution a second-order finite-difference discretization was shown to have good accuracy already for moderate discretizations. Since dense numerical linear algebra was used for the verification of the SSC they were not checked for the finest discretization used but still show a very consistent trend: the smallest eigenvalue of the projected Hessian of the Lagrangian suitably scaled by the discretization parameter behaves nearly constant indicating that the computed stationary solution appears to be a strict local minimizer. Subsequently, the procedure is applied to another parabolic and ten elliptic control problems, confirming the SSC in each case.

Open questions that should be addressed in future work include the following: can a formal proof be given of the satisfaction of the SSC which were numerically verified above? Can the results be generalized to the singular and bang-bang controls observed in [12]–[14]? Which numerical results can be obtained for state-constrained parabolic control problems? It is to be expected that such PDE-constrained optimization problems as considered in this paper will be subject of intense research efforts in the near future. We refer again to [16]; see also the short survey [8].

$1/h$	γ_h	γ_h/h^2
40	1.136e-3	1.82
50	7.447e-4	1.86
60	5.399e-4	1.94

Table 14: Minimal eigenvalue for (ED-5)

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