Optimization Techniques for Solving Elliptic Control Problems with Control and State Constraints. Part 2: Distributed Control

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Abstract: Part 2 continues the study of optimization techniques for elliptic control problems subject to control and state constraints and is devoted to distributed control. Boundary conditions are of mixed Dirichlet and Neumann type. Necessary conditions of optimality are formally stated in form of a local Pontryagin minimum principle. By introducing suitable discretization schemes, the control problem is transcribed into a nonlinear programming problem. The problems are formulated as AMPL [13] scripts and several optimization codes are applied. In particular, it is shown that a recently developed interior point method is able to solve these problems even for high discretizations. Several numerical examples with Dirichlet and Neumann boundary conditions are provided that illustrate the performance of the algorithm for different types of controls including bang-bang controls. The necessary conditions of optimality are checked numerically in the presence of active control and state constraints.

Key words: Elliptic control problems, distributed control, control and state constraints, discretization techniques, NLP-methods

1 Introduction

In Part 1 [23] of this paper we have studied discretization and optimization techniques for solving elliptic control problems with boundary control. Part 2 is devoted to optimization methods for elliptic control problems with distributed control. Boundary conditions are of mixed Dirichlet and Neumann type. The control process is subject to general control and state inequality constraints. Essentially the same discretization and optimization techniques as in Part 1 can be used for distributed control, the only difference being the fact that for distributed control the number of optimization variables roughly doubles compared to that for boundary control.

For the class of elliptic problems considered in this paper, necessary optimality conditions have not yet been derived to full extent in the literature. Necessary conditions
for more special classes of elliptic problems may be found in Bonnans, Casas [4, 5, 6]; cf. also Bergounioux et al. [3], Cañada et al. [8], Hettich et al. [17, 18], Leung, Stojanovic [21], Lions [22] and Rotin [27]. In section 2 we give a formal extension of the existing necessary conditions to the general elliptic problem treated in this paper. The conditions are specialized to cost functionals of tracking type and box constraints for control and state variables. In particular, a more detailed discussion is given for bang–bang and singular controls. In section 3 we show that the necessary conditions are consistent with their counterparts for the discretized problem which are obtained from the Kuhn–Tucker conditions.

In section 3, we discuss a full discretization approach in which both control and state variables are discretized. The resulting large-scale NLP–problem solved subsequently contains up to 80,000 variables. Six examples are considered in section 4. Optimal control and state as well as adjoint variables are computed which allow to check the discretized version of the necessary conditions with high accuracy. In Examples 2 and 5 we obtain an optimal bang–bang control which is especially remarkable in the distributed control example for which we are not aware of another instance in the literature.

The problems are formulated as AMPL [13] scripts and several optimization codes are applied. In particular, the interior point code LOQO [29] successfully and efficiently solved all problems. It needs to be stressed that more nonlinear formulations of these or other applications can be treated in the same way. The code LOQO was designed to solve general nonconvex NLP problems. For a comparison with other codes on several classes of optimization problems, see the benchmarks of [24].

2 Necessary conditions for elliptic control problems with control and state constraints

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary $\Gamma = \partial \Omega$. The derivative in the direction of the outward unit normal $\nu$ of $\Gamma$ will be denoted by $\partial_{\nu}$. Suppose that the boundary is partitioned as $\Gamma = \Gamma_1 \cup \Gamma_2$ with disjoints sets $\Gamma_1, \Gamma_2 \subset \Gamma$ that are composed of finitely many smooth and connected components.

We consider the optimal control problem of determining a distributed control function $u \in L^\infty(\Omega)$ that minimizes the functional

$$F(y, u) = \int_{\Omega} f(x, y(x), u(x)) \, dx + \int_{\Gamma_1} g(x, y(x)) \, dx$$

subject to the elliptic state equation,

$$-\Delta y(x) + d(x, y(x), u(x)) = 0, \quad \text{for } x \in \Omega,$$

Neumann and Dirichlet boundary conditions,

$$\partial_{\nu} y(x) = b(x, y(x)), \quad \text{for } x \in \Gamma_1,$$

$$y(x) = \gamma_1(x), \quad \text{for } x \in \partial \Omega.$$
\[ y(x) = y_2(x), \quad \text{for } x \in \Gamma_2, \]  
and mixed control–state inequality constraints, resp. \textit{pure} state inequality constraints,

\[
\begin{align*}
C(x, y(x), u(x)) &\leq 0, \quad \text{for } x \in \Omega, \quad (2.5) \\
S(x, y(x)) &\leq 0, \quad \text{for } x \in \Omega \cup \Gamma_1. \quad (2.6)
\end{align*}
\]

The split boundary formulation permits simultaneous treatment of various boundary conditions while in [23] this was done in separate sections. The functions \( f : \Omega \times \mathbb{R}^2 \to \mathbb{R}, \) \( g : \Gamma_1 \times \mathbb{R} \to \mathbb{R}, \) \( d : \Omega \times \mathbb{R}^2 \to \mathbb{R}, \) \( b : \Gamma_1 \times \mathbb{R} \to \mathbb{R}, \) \( C : \Omega \times \mathbb{R}^2 \to \mathbb{R}, \) and \( S : \Omega \cup \Gamma_1 \times \mathbb{R} \to \mathbb{R} \) are supposed to be \( C^1 \)-functions and \( y_2 \in C^1(\Gamma_2) \) is assumed in the Dirichlet condition (2.4). We have to admit that no numerical example with a splitting of the boundary \( \Gamma \) into \( \Gamma_1, \Gamma_2 \) will be considered in this paper. However, the splitting has been introduced to allow for a general discussion of necessary conditions. A practical example with splitted boundary may be found in [25].

The Laplacian \(-\Delta\) in the elliptic equation (2.2) can be replaced by an elliptic operator \( A \) in divergence form. We refer to section 2 of [23] for a precise definition. The above described control problem is slightly more general than the one considered in Bonnans and Casas [6] where first order conditions have been given in terms of a weak and strong Pontryagin principle. For \textit{linear} elliptic equations, first order conditions may also be found in Bergounioux et al. [3], Bonnans and Casas [5]. Nonlinear elliptic equations of Lotka–Volterra type have been considered in Cañada et al. [8] and Leung, Stojanovic [21, 28].

Throughout this paper, it will be assumed that an optimal solution \( \bar{u} \) and \( \bar{y} \) of problem (2.1)–(2.6) exists. To ensure well–posedness of the elliptic problem (2.1)–(2.3) we require as in Bonnans, Casas [6], condition (2.3), that

\[
d_{y}(x, y, u) \geq 0 \quad \forall (x, y, u) \in V, \quad (2.7)
\]

holds where \( V \) is a suitable bounded set containing the graph of the optimal solution. In special cases, such as Examples 1 and 2 below, one can dispense with this condition since well-posedness follows from special results cf. Gunzburger et al. [15]. However, we should note that condition (2.7) is \textit{not} satisfied for all numerical examples in section 4.

The \textit{active sets} for the inequality constraints (2.5), (2.6) are given by

\[
J(C) := \{ x \in \Omega \mid C(x, \bar{y}(x), \bar{u}(x)) = 0 \},
\]

\[
J(S) := \{ x \in \Omega \cup \Gamma_1 \mid S(x, \bar{y}(x)) = 0 \}. \quad (2.8)
\]

We do not study regularity conditions in detail and require that the following ones hold:

\[
C_u(x, \bar{y}(x), \bar{u}(x)) \neq 0 \quad \forall x \in J(C),
\]

\[
S_y(x, \bar{y}(x)) \neq 0 \quad \forall x \in J(S). \quad (2.9)
\]

3
Extending in a purely formal way the first order necessary conditions in Bonnans and Casas [6], we arrive at the following. There exists an adjoint state \( \bar{q} \in W^{1,1}(\Omega) \), a multiplier \( \tilde{\lambda} \in L^{\infty}(\Omega) \), and a regular Borel measure \( \tilde{\mu} \) in \( \Omega \) such that the following conditions hold:

**adjoint equation and boundary conditions:**

\[
-\Delta \bar{q}(x) + \bar{q}(x) d_y(x, \bar{y}(x), \bar{u}(x)) + f_y(x, \bar{y}(x), \bar{u}(x)) + \\
\quad \tilde{\lambda}(x) C_y(x, \bar{y}(x), \bar{u}(x)) + S_y(x, \bar{y}(x)) \tilde{\mu} = 0 \quad \text{in} \quad \Omega, \\
\partial_y \bar{q}(x) - \bar{q}(x) b_y(x, \bar{y}(x)) + f_g(x, \bar{y}(x)) \\
\quad + S_y(x, \bar{y}(x)) \tilde{\mu} = 0 \quad \text{on} \quad \Gamma_1, \\
\bar{q}(x) = 0 \quad \text{on} \quad \Gamma_2.
\]  

(2.10)

**minimum condition for** \( x \in \Omega \):

\[
f_u(x, \bar{y}(x), \bar{u}(x)) + \bar{q}(x) d_u(x, \bar{y}(x), \bar{u}(x)) + \tilde{\lambda}(x) C_u(x, \bar{y}(x), \bar{u}(x)) = 0.
\]  

(2.13)

**complementarity conditions:**

\[
\tilde{\lambda}(x) \geq 0 \quad \text{in} \quad J(C), \quad \tilde{\lambda}(x) = 0 \quad \text{in} \quad \Omega \setminus J(C), \\
d\tilde{\mu} \geq 0 \quad \text{in} \quad J(S), \quad d\tilde{\mu} = 0 \quad \text{in} \quad (\Omega \cup \Gamma_1) \setminus J(S).
\]  

(2.14)

The adjoint equations (2.10)–(2.12) are understood in the weak sense. According to Bourbaki [7], Chapter 9, the regular Borel measure in the adjoint equations (2.10) and (2.11) possesses a decomposition

\[
\tilde{\mu} = \bar{\nu} \cdot dx + \bar{\nu}_s \cdot \tilde{\mu}_s,
\]  

(2.15)

where \( dx \) represents the Lebesgue measure, the measure \( \tilde{\mu}_s \) is singular with respect to \( dx \) and \( \bar{\nu}, \bar{\nu}_s \) are measurable.

In many applications, the control and state constraints (2.5) and (2.6) are simple box constraints of the type

\[
u_1(x) \leq u(x) \leq u_2(x), \quad y(x) \leq \psi(x) \quad \text{a.e.} \quad x \in \Omega,
\]  

(2.16)

with functions \( \psi \in C(\overline{\Omega}) \) and \( u_1, u_2 \in L^{\infty}(\Omega) \). In this case, the adjoint equation (2.10) reduces to

\[
-\Delta \bar{q}(x) + \bar{q}(x) d_y(x, \bar{y}(x), \bar{u}(x)) + f_y(x, \bar{y}(x), \bar{u}(x)) + \tilde{\mu} = 0 \quad \text{in} \quad \Omega,
\]  

(2.17)

while the the minimum condition (2.13) yields the control law

\[
[f_u(x, \bar{y}(x), \bar{u}(x)) + \bar{q}(x) d_u(x, \bar{y}(x), \bar{u}(x))](u - \bar{u}(x)) \geq 0 \\
\quad \forall u \in [u_1(x), u_2(x)], \quad x \in \Omega.
\]  

(2.18)
A further specialization refers to a cost functional (2.1) of tracking type which has been considered frequently, cf. [1, 2, 19, 20],

$$F(y, u) = \frac{1}{2} \int_{\Omega} (y(x) - y_d(x))^2 \, dx + \frac{\alpha}{2} \int_{\Omega} (u(x) - u_d(x))^2 \, dx,$$

(2.19)

with given functions $y_d \in C(\Omega)$, $u_d \in L^\infty(\Omega)$, and a nonnegative weight $\alpha \geq 0$. Furthermore, let the function $d(x, y, u)$ in the state equation (2.2) be linear in the control variable with $d(x, y, u) = d_0(x, y) - u$. Then we deduce from (2.18) the minimum condition

$$[\alpha(\bar{u}(x) - u_d(x)) - \bar{q}(x)](u - \bar{u}(x)) \geq 0 \quad \forall u \in [u_1(x), u_2(x)], \text{ a.e. in } \Omega.$$  (2.20)

**Case $\alpha > 0$:** The previous conditions show that the optimal control $\bar{u}(x)$ is the projection of $u_d(x) + \bar{q}(x)$ onto the interval $[u_1(x), u_2(x)]$. More precisely, we have for $x \in \Omega$:

$$\bar{u}(x) = \begin{cases} u_d(x) + \bar{q}(x)/\alpha, & \text{if } u_d(x) + \bar{q}(x)/\alpha \in (u_1(x), u_2(x)), \\ u_1(x), & \text{if } u_d(x) + \bar{q}(x)/\alpha \leq u_1(x), \\ u_2(x), & \text{if } u_d(x) + \bar{q}(x)/\alpha \geq u_2(x). \end{cases}$$

(2.21)

**Case $\alpha = 0$:** We obtain an optimal control of bang-bang or singular type,

$$\bar{u}(x) = \begin{cases} u_1(x), & \text{if } \bar{q}(x) < 0, \\ u_2(x), & \text{if } \bar{q}(x) > 0, \\ \text{singular}, & \text{if } \bar{q}(x) = 0 \text{ on } \Omega_s \subset \Omega, \text{ meas}(\Omega_s) > 0. \end{cases}$$

(2.22)

### 3 Discretization and optimization techniques

#### 3.1 Discretization approach

The discussion of discretization schemes is restricted to the standard situation where the domain is the unit square $\Omega = (0, 1) \times (0, 1)$. The purpose of this section is to develop discretization techniques by which the distributed control problem (2.1)–(2.6) is transformed into a nonlinear programming problem (NLP-problem) of the form

Minimize $F^h(z)$ subject to $G^h(z) = 0$, $H(z) \leq 0$.  

(3.1)

The functions $F^h, G^h$ and $H$ are sufficiently smooth and are of appropriate dimension. The upper subscript $h$ denotes the dependence on the stepsizes. The optimization variable $z$ will comprise both the discretized state and control variables.

The form (3.1) will be achieved by solving the elliptic equation (2.2) with the standard five-point-star discretization scheme. Choose a number $N \in \mathbb{N}_+$ and the stepsizes $h := 1/(N + 1)$. Consider the mesh points

$$x_{ij} = (ih, jh), \quad 0 \leq i, j \leq N + 1,$$
and define the following sets of indices \((i, j)\) residing either in the domain \(\Omega\) or on the boundary \(\Gamma\), resp. on the subsets \(\Gamma_1, \Gamma_2\) of the boundary:

\[
I(\Omega) := \{ (i, j) \mid 1 \leq i, j \leq N \},
I(\Gamma) := \{ (i, j) \mid i, j = 0 \text{ or } i = N+1, \, j = 1, \ldots, N \}
\]

\[
I(\Gamma_k) := \{ (i, j) \in I(\Gamma) \mid x_{ij} \in \Gamma_k \}, \quad k = 1, 2,
I(\Omega \cup \Gamma_1) := I(\Omega) \cup I(\Gamma_1).
\] (3.2)

Obviously, we have \(\# I(\Omega) = N^2\), \(\# I(\Gamma) = 4 \cdot N\); define further \(M_1 := \# I(\Gamma_1)\).

Now we shall present discretization schemes for the distributed control problem that are similar to those for boundary controls considered in Part 1 [23]. The optimization variable \(z\) in (3.1) is taken as the vector

\[
z := \left( \left( y_{ij}, u_{ij} \right) \right)_{(i, j) \in I(\Omega \cup \Gamma_1)} \in \mathbb{R}^{2N^2 + M_1}.
\]

The remaining state variables \(y_{ij}, (i, j) \in I(\Gamma_2)\), are determined by the Dirichlet condition (2.4) as

\[
y_{ij} = y_2(x_{ij}) \quad \text{for} \quad (i, j) \in I(\Gamma_2).
\] (3.3)

The derivative \(\partial_r y(x_{ij})\) in the direction of the outward normal is approximated by the expression \(y_{ij}^\nu / \Delta t\) where

\[
y_{ij}^\nu := \begin{cases} y_{i0} - y_{i1}, & \text{for } j = 0, \quad i = 1, \ldots, N \\ y_{i0} - y_{i1}, & \text{for } i = 0, \quad j = 1, \ldots, N \\ y_{iN+1,j} - y_{iN,j}, & \text{for } i = N + 1, \quad j = 1, \ldots, N \\ y_{iN+1,j} - y_{iN}, & \text{for } j = N + 1, \quad i = 1, \ldots, N \end{cases}.
\] (3.4)

Then the discrete form of the Neumann boundary condition (2.3) leads to the equality constraints

\[
B^h(z) := y_{ij}^\nu - h b(x_{ij}, y_{ij}) = 0 \quad \text{for} \quad (i, j) \in I(\Gamma_1).
\] (3.5)

The application of the five–point–star to the elliptic equation \(-\Delta y(x) + d(x, y(x), u(x)) = 0\) in (2.2) yields the following equality constraints for all \((i, j) \in I(\Omega)\):

\[
G^h_{ij}(z) := 4y_{ij} - y_{i+1,j} - y_{i-1,j} - y_{i,j+1} - y_{i,j-1} + h^2 d(x_{ij}, y_{ij}, u_{ij}) = 0.
\] (3.6)

Note that the Dirichlet condition (3.3) is used in this equation to substitute the variables \(y_{ij}\) for \((i, j) \in I(\Gamma_2)\). The control and state inequality constraints (2.5) and (2.6) yield the inequality constraints

\[
C(x_{ij}, y_{ij}, u_{ij}) \leq 0, \quad \forall \; (i, j) \in I(\Omega), \quad \text{and} \quad S(x_{ij}, y_{ij}) \leq 0, \quad \forall \; (i, j) \in I(\Omega \cup \Gamma_1).
\] (3.7) \quad (3.8)
Observe that these inequality constraints do not depend on the meshsize \( h \). The discretized form of the cost function (2.1) is

\[
F^h(z) := h^2 \sum_{(i,j) \in I(\Omega)} f(x_{ij}, y_{ij}, u_{ij}) + h \sum_{(i,j) \in I(\Gamma_1)} g(x_{ij}, y_{ij}).
\]  

(3.9)

In summary, the relations (3.5)–(3.9) define a NLP–problem of the form (3.1). The corresponding Lagrangian function is

\[
L(z, q, \lambda, \mu) := h^2 \sum_{(i,j) \in I(\Omega)} f(x_{ij}, y_{ij}, u_{ij}) + h \sum_{(i,j) \in I(\Gamma_1)} g(x_{ij}, y_{ij})
+ \sum_{(i,j) \in I(\Omega)} [q_{ij} G^h_{ij}(z) + \lambda_{ij} C(x_{ij}, y_{ij}, u_{ij}) + \mu_{ij} S(x_{ij}, y_{ij})]
+ \sum_{(i,j) \in I(\Gamma_1)} \left[ \mu_{ij} S(x_{ij}, y_{ij}) + q_{ij} B^h(z) \right],
\]  

(3.10)

where the Lagrange multipliers \( q = (q_{ij})_{(i,j) \in I(\Omega \cup \Gamma_1)}, \lambda = (\lambda_{ij})_{(i,j) \in I(\Omega)} \) and \( \mu = (\mu_{ij})_{(i,j) \in I(\Omega \cup \Gamma_1)} \) are associated with the equality constraints (3.5) and (3.6), resp. the inequality constraints (3.7) and (3.8). In addition, the multipliers \( \lambda \) and \( \mu \) satisfy the complementarity conditions corresponding to (2.14):

\[
\lambda_{ij} \geq 0 \quad \text{and} \quad \lambda_{ij} C(x_{ij}, y_{ij}, u_{ij}) = 0 \quad \forall (i, j) \in I(\Omega),
\]

\[
\mu_{ij} \geq 0 \quad \text{and} \quad \mu_{ij} S(x_{ij}, y_{ij}) = 0 \quad \forall (i, j) \in I(\Omega \cup \Gamma_1).
\]

In the next step we discuss the necessary conditions of optimality:

\[
0 = L_z = \left( (L_{y_{ij}})_{(i,j) \in I(\Omega \cup \Gamma_1)}, (L_{u_{ij}})_{(i,j) \in I(\Omega)} \right).
\]

For state variables \( y_{ij} \) with indices \( (i, j) \in I(\Omega) \) we obtain the relations

\[
0 = L_{y_{ij}} = d_{ij} - q_{i+1,j} - q_{i-1,j} - q_{i,j+1} - q_{i,j-1} + h^2 q_{ij} d_y(x_{ij}, y_{ij}, u_{ij}) + h^2 f_y(x_{ij}, y_{ij}, u_{ij}) + \lambda_{ij} C_y(x_{ij}, y_{ij}, u_{ij}) + \mu_{ij} S_y(x_{ij}, y_{ij}).
\]  

(3.11)

In these equations, the up to now undefined multipliers are set to

\[
q_{ij} = 0 \quad \forall (i, j) \in \Gamma_2,
\]  

(3.12)

which is in accordance with the Dirichlet condition (2.12). We deduce from equations (3.11) that the Lagrange multipliers \( q = (q_{ij}) \) satisfy the five–point–star difference equations for the adjoint equation \( -\Delta \bar{q} + \bar{q} d_y + f_y + \bar{\lambda} C_y + S_y \bar{\mu} = 0 \) in (2.10) if we use the following approximations for the multiplier function \( \bar{\lambda} \) and the regular Borel measure \( \bar{\mu} \),

\[
\bar{\lambda}(x_{ij}) \sim \lambda_{ij}/h^2, \quad \int_{sq(h^2)} d\bar{\mu} \sim \mu_{ij},
\]  

(3.13)

where in the second relation \( sq(h^2) \) denotes a square centered at \( x_{ij} \) with area \( h^2 \). Recall the decomposition (2.15) of the measure \( \bar{\mu} = \bar{\nu} \cdot dx + \bar{\nu}_s \cdot \bar{\mu}_s \). If the singular
part of the measure vanishes, i.e. \( \nu_s \cdot \bar{\nu}_s = 0 \) holds, then (3.13) yields the following approximation for the density \( \bar{\nu} \),

\[
\bar{\nu}(x_{ij}) \sim \mu_{ij}/h^2.
\]

In case that the measure \( \mu = \nu_s \cdot \delta(x - x_{ij}) \) is a delta distribution, we obtain from (3.13) the approximation

\[
\nu_s \sim \mu_{ij}.
\]

For indices \( (i,j) \in I(\Gamma) \) on the boundary \( \Gamma \), e.g., for \( j = 0, i \in \{1,\ldots,N\} \), we obtain

\[
0 = L_{y_{i0}} = -q_{i0} + q_{i0} h b_y(x_{i0}, y_{i0}) + h g_y(x_{i0}, y_{i0}) + \mu_{ij} S_y(x_{ij}, y_{ij}).
\]

These relations represent the discrete version of the Neumann boundary condition (2.11) if we approximate the regular Borel measure \( \mu \) on the boundary by

\[
\int_{s(h)} d\mu \sim \mu_{ij},
\]

where \( s(h) \) denotes a line segment of length \( h \) on \( \Gamma \) centered at \( x_{ij} \). If the singular part in the decomposition (2.15) vanishes resp. if the measure \( \bar{\mu} = \nu_s \cdot \delta(x - x_{ij}) \) is a delta distribution, we obtain the following approximations

\[
\bar{\nu}(x_{ij}) \sim \mu_{ij}/h \quad \text{resp.} \quad \nu_s \sim \mu_{ij}.
\]

Finally, necessary conditions with respect to the control variables \( u_{ij} \) for \( (i,j) \in I(\Omega) \) are determined by

\[
0 = L_{u_{ij}} = h^2 f_u(x_{ij}, y_{ij}, u_{ij}) + q_{ij} h^2 d_u(x_{ij}, y_{ij}, u_{ij}) + \lambda_{ij} C_u(x_{ij}, y_{ij}, u_{ij}).
\]

From this equation we recover the discrete version of the control law (2.13), if we use the same identification \( \lambda(x_{ij}) \sim \lambda_{ij}/h^2 \) as in (3.13).

### 3.2 Optimization codes and modeling environment

As in part 1 of this paper deliberately standard NLP software was utilized in conjunction with a modeling language to numerically solve the large discrete optimization problems. Modeling languages such as AMPL [13] which will be used below permit the formulation of the problem in a language particularly suitable for this purpose. Subsequently, a number of solvers, written in different programming languages, may be called through an interface which, in the case of AMPL, is provided for free. Below the following solvers will be used: LANCELOT A [12], MINOS-5.5 [26], SNOPT-5.3-4 [14], and LOQO-4.01 [29]. Especially, the only interior point code LOQO proved to be robust and efficient for the type of problems considered. An important feature of AMPL is its automatic differentiation capability. Only function values need to be provided for objective and constraint functions.
4 Numerical examples

We discuss numerical solutions for elliptic problems with the following specifications: the domain is the unit square $\Omega = (0, 1) \times (0, 1)$ and the control and state constraints are box constraints of the form (2.16). For each example a table is given listing the following for different grid sizes: Number of iterations and CPU time in seconds needed by LOQO (including AMPL), the number of accurate digits in the computed optimal objective function value, and this number. More detailed results including a comparison with the other solvers mentioned are exemplarily given for Examples 4 and 5.

4.1 Example 1

In this example we choose a nonlinear partial differential equation and homogeneous Dirichlet boundary conditions:

\[
\begin{align*}
&\text{on } \Omega: \quad -\Delta y(x) - y(x) + y(x)^3 = u, \quad y(x) \leq .185, \quad 1.5 \leq u(x) \leq 4.5, \\
&\quad y_d(x) = 1 + 2[x_1(x_1 - 1) + x_2(x_2 - 1)], \\
&\text{on } \Gamma: \quad y(x) = 0, \\
&\quad u_d(x) \equiv 0, \quad \alpha = 0.001.
\end{align*}
\]

These equations are related to a simplified Ginzburg–Landau model for superconductivity in the absence of internal magnetic fields with $y$ the wave function; cf. Ito, Kunisch [19] and Kunisch, Volkwein [20].

The adjoint equations (2.10), (2.12) apply:

\[
-\Delta \tilde{q}(x) + \tilde{q}(x)(-1 + 3\tilde{y}(x)^2) + \tilde{y}(x) - y_d(x) + \bar{\mu} = 0 \quad \text{in} \quad \Omega, \\
\tilde{q}(x) = 0 \quad \text{on} \quad \Gamma.
\]

(4.1)

Since $u_d = 0$, the minimum condition (2.21) yields the control law

\[
u(x) = P_{[u_1, u_2]}(\tilde{q}(x)/\alpha),
\]

(4.2)

where $P_{[u_1, u_2]}$ denotes the projection operator onto the interval $[u_1, u_2] = [1.5, 4.5]$. The optimal control is shown in Figure 1, while Figure 2 depicts the optimal state and associated adjoint variable. The adjoint variable permits the verification of the control law (4.2). The state is active in the center $(0.5, 0.5)$. Figure 2 indicates that condition (2.7) is not satisfied since $d_y(x, \tilde{y}(x)) = 3\tilde{y}(x)^2 - 1 < 0$ holds for all $x \in \Omega$. However, well-posedness of the Ginzburg–Landau model follows from results in Gunzburger et al. [15].

4.2 Example 2

We choose $\alpha = 0$ in Example 1 and expect to obtain a bang–bang control. The adjoint equation agrees with that in (4.1). In view of $\alpha = 0$ the control law (2.22) yields

\[
\bar{u}(x) = \begin{cases} 
  u_1 = 1.5, & \text{if } \quad \tilde{q}(x) < 0 \\
  u_2 = 4.5, & \text{if } \quad \tilde{q}(x) > 0 
\end{cases}
\]

(4.3)
Figure 1: Example 1, $\alpha = 0.001$ : Optimal control.

Figure 2: Example 1, $\alpha = 0.001$ : Optimal state and adjoint variable
Figure 3: Example 2, $\alpha = 0$ : Optimal control and switching curve $\bar{q}(x) = 0$.

Figure 4: Example 2, $\alpha = 0$ : Optimal state and adjoint variable
\[
\begin{array}{|c|c|c|c|c|}
\hline
N+1 & \text{it} & \text{CPU} & \text{Acc} & F(\bar{y}) \\
\hline
50 & 26 & 104 & 8 & .0577903 \\
100 & 30 & 1897 & 8 & .0621615 \\
200 & 35 & 54831 & 8 & .0644259 \\
\hline
\end{array}
\]

Table 1: Information on solution of Example 1

\[
\begin{array}{|c|c|c|c|c|}
\hline
N+1 & \text{it} & \text{CPU} & \text{Acc} & F(\bar{y}) \\
\hline
50 & 30 & 112 & 8 & .0521138 \\
100 & 37 & 2169 & 8 & .0564474 \\
200 & 45 & 67769 & 8 & .0596968 \\
\hline
\end{array}
\]

Table 2: Information on solution of Example 2

Figure 3 shows that the optimal control is indeed bang–bang and does not exhibit singular parts. The optimal state and adjoint variables are displayed in Figure 4. Both figures allow to verify precisely the switching conditions of the control law (4.3).

### 4.3 Example 3

In this example we choose another nonlinear partial differential equation and homogeneous Dirichlet boundary conditions:

on \( \Omega \):

\(-\Delta y(x) - \exp(y(x)) = u, \quad y(x) \leq .11, \quad -5 \leq u(x) \leq 5, \)

\(y_d(x) = \sin(2\pi x_1) \sin(2\pi x_2),\)

on \( \Gamma \):

\(y(x) = 0, \quad u_d(x) \equiv 0, \quad \alpha = 0.001.\)

This problem has been considered in [16], but without state constraints.

\[
\begin{array}{|c|c|c|c|c|}
\hline
N+1 & \text{it} & \text{CPU} & \text{Acc} & F(\bar{y}) \\
\hline
50 & 29 & 131 & 8 & .110242 \\
100 & 32 & 2257 & 8 & .110263 \\
200 & 31 & 42644 & 8 & .110269 \\
\hline
\end{array}
\]

Table 3: Information on solution of Example 3

The adjoint equations (2.10), (2.12) yield

\[
-\Delta \bar{q}(x) - \bar{q}(x) \exp(\bar{g}(x)) + \bar{g}(x) - y_d(x) + \bar{\mu} = 0 \quad \text{in} \quad \Omega, \\
\bar{q}(x) = 0 \quad \text{on} \quad \Gamma. \quad (4.4)
\]
Figure 5: Example 3, $\alpha = 0.001$ : Optimal control.

Figure 6: Example 3, $\alpha = 0.001$ : Optimal state and adjoint variable
The minimum condition (2.21) leads again to the projection

\[ \tilde{u}(x) = P_{[u_1,u_2]} \left( \bar{q}(x)/\alpha \right) \]  

(4.5)

with \([u_1,u_2] = [-5,5]\). The optimal control is shown in Figure 5, while Figure 6 displays the optimal state and associated adjoint variable. The adjoint variable allows to verify the control rule (4.5). Note that condition (2.7) does not hold for this example in view of \( d_y(x, \bar{y}(x)) = -\exp(\bar{y}(x)) < 0 \).

The state is active in the points \( x^1 = (0.25, 0.25), x^2 = (0.75, 0.75) \). In view of the decomposition (2.15), the measure \( \bar{\mu} \) is given here by the singular measure \( \bar{\mu} = \bar{p}_1 \delta(x-x^1) + \bar{p}_2 \delta(x-x^2) \), where \( \bar{p}_1, \bar{p}_2 \) are identified with the multipliers \( \mu_{ij} \) in view of (3.15). Let us check now the accuracy with which the computed adjoint equation is satisfied. At the point \( x_{ij} = x^1 = (0.25, 0.25) \) we obtain \( q_{ij} = 0.0085852, q_{i+1,j} = 0.0092638, q_{i-1,j} = 0.0091208, q_{i,j+1} = 0.0092638, q_{i,j-1} = 0.0091208, y_{ij} = 0.11, y_d(x_{ij}) = 1, \mu_{ij} = 0.0025181 \). Then the discretized adjoint equation (3.11) yields \( 4q_{ij} - q_{ij} - q_{i+1,j} - q_{i-1,j} - q_{i,j+1} - q_{i,j-1} - h^2 q_{ij} \exp(y_{ij}) + h^2 (y_{ij} - y_d(x_{ij})) + \mu_{ij} = -0.0000003 \) for \( h = 1/100 \).

4.4 Example 4

In this and the following examples we choose Neumann boundary conditions. We modify Example 3 and consider the problem:

on \( \Omega : \quad -\Delta y(x) - \exp(y(x)) = u, \quad y(x) \leq .371, \quad -8 \leq u(x) \leq 9, \)
\[ y_d(x) = \sin(2\pi x_1) \sin(2\pi x_2), \]
on \( \Gamma : \quad \partial_n y(x) + y(x) = 0, \quad u_d(x) \equiv 0, \alpha = 0.001. \)

The adjoint equation agrees with (4.4) except that we obtain a Neumann boundary condition

\[ \partial_n \bar{q}(x) + \bar{q}(x) = 0 \quad \text{on} \ \Gamma. \]  

(4.6)

The minimum condition (2.21) gives the projection (4.5) with \([u_1,u_2] = [-8,9]\). Figure 7 shows the optimal control while Figure 8 depicts the optimal state and associated adjoint variable. The reader may verify that the control rule (4.5) is satisfied.

This example will also be used to illustrate the numerical process in more detail. The platform is a 450 MHz Pentium-II PC with Linux-2.2.12. Table 4 summarizes the data: size of grid, number of iterations of LOQO, times in seconds for the AMPL compilation and the solution by LOQO, the accuracy as the number of correct significant digits in the discrete objective function value only. The primal–dual approach yields upper and lower bounds for this value and thus permits such a statement. In all the examples solved this accuracy measure was at least 8 showing that LOQO had converged satisfactorily. However, from the accuracy of the objective function value no
Figure 7: Example 4, $\alpha = 0.001$ : Optimal control.

Figure 8: Example 4, $\alpha = 0.001$ : Optimal state and adjoint variable
similar accuracy of the computed solution components may in general be inferred. In fact, for the discretizations used they may be expected to have at most a few correct digits. Finally, we list the objective function value and the value in the center of the domain of both the state and the control variable. These values show that only a moderate accuracy may be expected.

<table>
<thead>
<tr>
<th>N+1</th>
<th>it</th>
<th>AMPL</th>
<th>LOQO</th>
<th>Acc</th>
<th>$F(y, u)$</th>
<th>$y(5, 5)$</th>
<th>$u(5, 5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>26</td>
<td>.2</td>
<td>110</td>
<td>8</td>
<td>.0773888</td>
<td>-0.011311</td>
<td>-1.688123</td>
</tr>
<tr>
<td>100</td>
<td>23</td>
<td>1</td>
<td>1325</td>
<td>8</td>
<td>.0780638</td>
<td>-0.009160</td>
<td>-1.620104</td>
</tr>
<tr>
<td>200</td>
<td>25</td>
<td>4</td>
<td>43640</td>
<td>8</td>
<td>.0784259</td>
<td>-0.008243</td>
<td>-1.588789</td>
</tr>
</tbody>
</table>

Table 4: Detailed information on solution of Example 4

We compare LOQO and other solvers with AMPL interface available to us in Table 5; links to all codes in [24]. Given are CPU times in seconds on the same platform used above, but scaled by LOQO’s time. MINOS and SNOPT report the largest problem as infeasible while LANCELOT needs excessive compute time. Both SNOPT and MINOS are designed for problems with a moderate number of degrees of freedom, about 2000 and thus should not be expected to be able to handle the larger instances. See, however, the results for Example 5 which has a very small number of degrees of freedom, understood here as the difference between the number of variables and the number of active constraints. All solvers were given the same AMPL file and AMPL initializes variables with zero if no explicit initialization is made. In summary one can say that for $N+1 = 100$ the computing times for LOQO are acceptable and that the computed solutions, both function values and variables, appear to be in error by a few units in the third significant digit.

Table 5: Comparison of different solvers on Example 4

<table>
<thead>
<tr>
<th>N+1</th>
<th>LANCELOT</th>
<th>LOQO</th>
<th>MINOS</th>
<th>SNOPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>4.4</td>
<td>1</td>
<td>5.8</td>
<td>71</td>
</tr>
<tr>
<td>100</td>
<td>16</td>
<td>1</td>
<td>3.1</td>
<td>88</td>
</tr>
<tr>
<td>200</td>
<td>-</td>
<td>1</td>
<td>inf</td>
<td>inf</td>
</tr>
</tbody>
</table>

4.5 Example 5

This example has the same data as Example 4 except for $\alpha = 0$. Figure 9 indicates that the optimal control is bang-bang and satisfies the switching conditions inferred
Figure 9: Example 5, $\alpha = 0$ : Optimal control and state.

Figure 10: Example 5, $\alpha = 0$ : Adjoint variable and switching locus $\bar{q}(x) = 0$. 
from (2.22),

\[ \bar{u}(x) = \begin{cases} 
  u_1 = -8, & \text{if } \bar{q}(x) < 0 \\
  u_2 = 9, & \text{if } \bar{q}(x) > 0 
\end{cases} \]

SNOPT fails for larger \( N \) while LANCELOT needs excessive compute time. Figure 10 shows the associated adjoint variable and switching function.

<table>
<thead>
<tr>
<th>N+1</th>
<th>it</th>
<th>LOQO</th>
<th>Acc</th>
<th>( F(y) )</th>
<th>SNOPT</th>
<th>MINOS</th>
<th>LANCELOT</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>37</td>
<td>145</td>
<td>8</td>
<td>0.0539133</td>
<td>2399</td>
<td>205</td>
<td>923</td>
</tr>
<tr>
<td>100</td>
<td>60</td>
<td>5735</td>
<td>8</td>
<td>0.0544745</td>
<td>f</td>
<td>4432</td>
<td>33390</td>
</tr>
<tr>
<td>200</td>
<td>84</td>
<td>137514</td>
<td>8</td>
<td>0.0547818</td>
<td>f</td>
<td>119971</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6: Information on solution of Example 5

4.6 Example 6, an elliptic system of logistic type

In this section we consider an optimal control problem for a semilinear elliptic equation of logistic type which was studied in Leung, Stojanovic [21, 28]. The problem is to determine a distributed control \( u \in L^\infty(\Omega) \) that minimizes the functional

\[ F(y, u) = \int_\Omega (Mu(x)^2 - Ku(x)y(x)) \, dx \]  

(4.7)

subject to the elliptic state equation

\[ -\Delta y(x) = y(x)(a(x) - u(x) - by(x)), \quad \text{for } x \in \Omega, \]  

(4.8)

homogeneous Neumann boundary conditions,

\[ \partial_n y(x) + \beta(x)y(x) = 0, \quad \text{for } x \in \Gamma, \]  

(4.9)

and control and state inequality constraints

\[ u_1 \leq u(x) \leq u_2, \quad y(x) \leq \psi(x), \quad \text{for } x \in \Omega. \]  

(4.10)

Here, \( y(x) \) denotes the population of a biological species, \( a(x) \) a spatially dependent intrinsic growth rate, \( b \) the crowding effect, while \( F \) denotes the difference between economic cost and revenue, with nonnegative constants \( M, K \). In [21, 28] the function \( \beta(x) \) was chosen as 0. Numerical results for this case can be found in [25]. Here, we consider \( \beta(x) = 1 \) if \( x_1 = 0 \) or \( x_2 = 0 \) and \( \beta(x) = 0 \) otherwise. The goal is to find a control function which maximizes profit. A similar control problem with Dirichlet boundary conditions was recently studied by Cañada et al. [8].
Figure 11: Example 6 : Optimal control.

Figure 12: Example 6 : Optimal state and adjoint variable
The adjoint equations (2.10), (2.11) yield the following equations:

\[-\Delta \bar{q}(x) + \bar{q}(x) [2b \bar{y}(x) + \bar{u}(x) - a(x)] - K \bar{u}(x) + \bar{\mu} = 0, \quad \text{in } \Omega,\]

\[\partial_n \bar{q}(x) + \beta(x)\bar{q}(x) = 0, \quad \text{on } \Gamma.\]

For \( M > 0 \), the minimum condition (2.18) gives the control law

\[\bar{u}(x) = P_{[u_1, u_2]} \left( \frac{1}{2M} \left[ (K - \bar{q}(x)) \bar{y}(x) \right] \right),\] (4.11)

where \( P_{[u_1, u_2]} \) denotes the projection operator on the interval \([u_1, u_2]\).

The following concrete data were used:

\[a(x) = 7 + 4 \sin(2\pi x_1 x_2), \ b = 1, \ M = 1, \ K = 0.8,\]
\[u_1 = 1.4, \ u_2 = 1.6, \ \psi(x) = 6.09.\]

Figure 11 displays the optimal control while Figure 12 shows the optimal state and adjoint variable. The reader may verify that the minimum condition given by the projection (4.11) holds with high accuracy. However, note that condition (2.7) imposed in [6] is not satisfied everywhere in \( \Omega \).

<table>
<thead>
<tr>
<th>( N+1 )</th>
<th>it</th>
<th>CPU</th>
<th>Acc</th>
<th>( F(\bar{y}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>29</td>
<td>104</td>
<td>8</td>
<td>-4.19322</td>
</tr>
<tr>
<td>100</td>
<td>32</td>
<td>2235</td>
<td>8</td>
<td>-4.27569</td>
</tr>
<tr>
<td>200</td>
<td>33</td>
<td>42543</td>
<td>8</td>
<td>-4.31709</td>
</tr>
</tbody>
</table>

Table 7: Information on solution of Example 6

### 4.7 Conclusion

In continuation of the first part of this paper devoted to boundary control we have developed numerical techniques for solving semilinear distributed control problems with control and state constraints. While three numerical methods, two of interior point type, were compared in [1] for linear problems and homogeneous Dirichlet conditions the emphasis in this work is on treating nonlinearities in both the equations and the boundary equations, see part 1, which are of Dirichlet and Neumann type. The control problem is fully discretized resulting in a large, sparse nonlinear optimization (NLP) problem. Modern NLP software is utilized for its solution and the interior point code LOQO [29] proves to be a robust and efficient tool. While also results for several other NLP programs are given these are only meant to show what a straightforward application, with default options, of these to the problems at hand and through the common interface AMPL [13] yields. The algorithms used are
quite different and, for example, LOQO makes use of second derivatives while the quasi–Newton based SQP code SNOPT [14] does not. A total of six problems were solved. The necessary optimality conditions of section 3.1 were checked in all cases. They are given examplarily in one, example 3, with an exponential nonlinearity. In particular, bang-bang controls were computed in two cases and, to the best of our knowledge, for the first time in distributed two-dimensional control. We refer to the conclusion of part 1 concerning the planned generalizations of our work.

References


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