Rational Interpolation through the Optimal Attachment of Poles to the Interpolating Polynomial

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After recalling some pitfalls of polynomial interpolation (in particular slopes limited by Markov's inequality) and rational interpolation (e.g., unattainable points, poles in the interpolation interval, erratic behavior of the error for small numbers of nodes), we suggest an alternative for the case when the function to be interpolated is known everywhere, not just at the nodes. The method consists in replacing the interpolating polynomial with a rational interpolant whose poles are all prescribed, written in its barycentric form as in [4], and optimizing the placement of the poles in such a way as to minimize a chosen norm of the error.

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1. Introduction

In the present work we will address the problem of interpolating a given continuous function f between $N+1$ distinct points $x_0, x_1, \dots, x_N$ in an interval $[a,b]$. We may choose $[a,b] := [-1,1]$ without loss of generality.

The classical solution is the interpolating polynomial p of degree $\leq N$, whose determination is always a well-posed problem, but the use of which is merely reasonable for special choices of the x_k's, i.e., for points whose preimages on the circle by the application arccos are almost equidistant. As it is well known [16, p. 99], [12], for equidistant points on the interval the polynomials p diverge or are ill–conditioned as N increases.

But even with good points like Čebyšev's, polynomial interpolation may not be adequate. Markov's inequality states that $\|q_n'\|_\infty \le n^2\|q_n\|_\infty$ for every polynomial q_n of degree $\le n$. On the other hand, if q_n is a good approximation to f, then $\|q_n\|_\infty = \|f\|_\infty + e$, e small, and thus $\|q_n'\|_\infty \le n^2(\|f\|_\infty + e)$; therefore, no function f with a derivative much larger than $n^2\|f\|_\infty$ at some point can be well approximated by a polynomial of degree n (in the sense that its derivative is also approximated well). In other words, for a good approximation of f' as well as f the degree of the interpolating polynomial should be at least $\sqrt{\|f'\|_\infty/\|f\|_\infty}$. If $\|f'\|_\infty \gg \|f\|_\infty$, then a simultaneous "good" approximation of f and f' requires working with interpolating polynomials p of such a large degree that this may be numerically prohibitive.

The next infinitely differentiable choice for such functions as well as for arbitrary nodes seems to be rational interpolation [10,6]. The classical rational interpolant can be computed in a finite number of operations, but it has several drawbacks:

- at least for small N, it does not always exist and, if it does, it may exhibit poles in the interval of interpolation, which is not acceptable for the approximation of a continuous function (see Example 4 in §4);
- in general these disadvantages disappear as N increases, but when N becomes larger than about 100 the computation is often affected by instability as a consequence of smearing (see the examples for equidistant points in [6]).

As a way of approximating functions with large gradients we suggest here to replace the interpolating polynomial of degree $\le N$ with the quotient of a polynomial of degree $\le N$ and a polynomial of prescribed degree that diminishes the maximal error as much as possible.

2. Attaching poles to the interpolating polynomial

Let $\mathcal{P}_m$ and $\mathcal{R}_{m,n}$ respectively denote the linear space of all polynomials of degree $\le m$ and the set of all rational functions with numerator degree $\le m$ and denominator degree $\le n$; furthermore, denote by f_k the interpolated values $f(x_k)$, $k = 0(1)N$, of f. Then the unique polynomial $p \in \mathcal{P}_N$ that interpolates f between the x_k's can be written in its *barycentric form* [12]

$$p(x) = \sum_{k=0}^{N} \frac{w_k}{x - x_k} f_k \Bigg/ \sum_{k=0}^{N} \frac{w_k}{x - x_k}, \tag{1}$$

where the so-called *weight* w_k corresponding to the point x_k is given by

$$w_k := 1 \Big/ \prod_{i=0,\ i\neq k}^{N} (x_k - x_i).$$

The barycentric formula has several advantages [18], [6, p. 357]. One of them is the fact that the weights appear in the numerator and in the denominator, so that they can be divided by any common factor. For example, simplified weights for equidistant points are given by

$$w_k^* = (-1)^k \binom{N}{k}$$

[8], while for the Čebyšev points of the second kind $\cos\phi_k$, $\phi_k := k\frac{\pi}{N}$, one simply has [17]

$$w_k^* = (-1)^k \delta_k, \qquad \delta_k = \begin{cases} 1/2, & k = 0 \text{ or } k = N, \\ 1, & \text{otherwise.} \end{cases}$$

As explained in the introduction, we now want to improve the quality of approximation of the interpolant, for instance for functions with very large derivatives. For that purpose, we will divide the interpolant by an optimized denominator, while maintaining interpolation.

Let P, $P \leq N$, be the number of poles z_i, $i = 1(1)P$, we want to attach to the polynomial. If some rational interpolant $r \in \mathcal{R}_{N,P}$ exists with poles at the z_i's and only there, then its denominator takes the values

$$d_k := a \prod_{i=1}^{P} (x_k - z_i), \quad a \neq 0 \in \mathbb{C} \text{ arbitrary}, \tag{2}$$

at the interpolation points x_k. (The fact that it does not exist may mean that attaching the poles is not advisable from an approximation point of view, see [4].) To insure interpolation, the values of the numerator at the same points will be $f_k d_k$. Writing the numerator and the denominator as interpolating polynomials in their barycentric form (1) and simplifying, one gets

$$r := \sum_{k=0}^{N} \frac{w_k \prod\limits_{i=1}^{P} (x_k - z_i)}{x - x_k} f_k \Bigg/ \sum_{k=0}^{N} \frac{w_k \prod\limits_{i=1}^{P} (x_k - z_i)}{x - x_k}. \tag{3}$$

(3) is the barycentric representation of r with weights $v_k := w_k \prod_{i=1}^{P} (x_k - z_i)$. In the present case, with all poles prescribed, the weights are unique up to a cons-

tant [4]. Barycentric representations exist for every rational interpolant in $\mathcal{R}_{N,N}$ [5], and computing them is a way of solving the classical rational interpolation problem also when only a subset of the poles are prescribed [4].

In order to stay with real interpolants, we will assume here that the poles z_i with $\Im z_i \neq 0$ arise in complex conjugate pairs.

3. The optimization problem

In order to find a best denominator, we will solve the optimization problem of minimizing $\|r-f\|_\infty$ with respect to the z_i's, where r is given by (3).

It should be noted that it is not possible that a pole z_i thereby comes to lie in the interval $[-1,1]$: the corresponding r could never be a best approximation to the continuous f. In particular, no z_i can be 0.

The *existence* of an optimum is easily seen. For that purpose, write r as

$$r := \sum_{k=0}^{N} \frac{w_k \prod\limits_{i=1}^{P} (1 - \frac{x_k}{z_i})}{x - x_k} f_k \Bigg/ \sum_{k=0}^{N} \frac{w_k \prod\limits_{i=1}^{P} (1 - \frac{x_k}{z_i})}{x - x_k} \tag{4}$$

and consider every z_i on its Riemann sphere $\overline{\mathbb{C}}$. For the polynomial, every pole is at infinity, i.e., at the north pole of its sphere. And there must exist a set of z_i's for which $\|r-f\|_\infty$ is minimal, since the latter is a continuous function over the cross product of the P spheres, which is compact.

The *unicity* question is more involved. The constant function example $f \equiv c$ (and, more generally, every polynomial of degree less or equal to $N-P$) shows that there are $f \in C[a,b]$ for which no pole can be attached to p [4]: every set of numbers replacing the w_k in (1) results in $r \equiv f$ (which appears only reasonable from an approximation point of view). In practical computations, such a large connected set of optimal points manifests itself in the optimization routine going around without direction.

Unicity is warranted if the optimization yields $\|r-f\|_\infty = 0$, i.e., if $r = f$ on $[-1,1]$. Indeed, when two analytic functions r_1, r_2 agree on an interval I, then $r_1 = r_2$ on every domain of analyticity containing I, by the fundamental lemma on analytic continuation [13, p. 150]. Therefore $\|r-f\|_\infty = 0$ if and only if $f = r \in \mathcal{R}_{N,P}$ on $[-1,1]$. Numerically, however, the unicity shows up only if P is not chosen larger than the actual number of poles of f, for otherwise the condition of the problem is so good that many combinations of P poles yield $\|r-f\|_\infty \le \nu$, $\nu =$ unit-roundoff error of the machine.

The unicity question can also be narrowed to an interesting one, to which we do not have the answer. It is obvious from the above construction that nonvanishing of the numerator at z_i is a sufficient condition for r to have a pole there. We have checked this condition in its equivalent form [4]

$$c_i := \sum_{k=0}^{N} w_k f_k \prod_{\substack{j=1\\ j\neq i}}^{P} (x_k - z_j) \neq 0. \tag{5}$$

Do the conditions (5)—one for every z_i—imply that the corresponding optimal r is the only one minimizing $\|r-f\|_\infty$?

We want to point to another representation of (5). Indeed, $\sum_{k=0}^{N} w_k g(x_k)$ is the leading coefficient of the polynomial interpolating a function g between the x_k's, and this coefficient is the divided difference of g with respect to all x_k's [2]. Condition (5) can therefore be written as

$$g_i[x_0, x_1, \dots, x_N] \neq 0, \quad \text{where } g_i(x) := f(x) \prod_{\substack{j=1\\ j\neq i}}^{P} (x - z_j). \tag{6}$$

A nice property of the suggested interpolation deserves special notice: the approximation error cannot increase with the number of poles, this in sharp contrast with classical rational interpolation. Indeed, as a new unknown, say z_P, is added to the set of variables, $\{z_1, \dots, z_{P-1}\}$, the optimal value of the latter is a feasible vector for the higher dimensional optimization—simply set $z_P = \infty$ in (4). In particular, attaching poles to the interpolating polynomial can never worsen the quality of the approximation.

4. Numerical experiments

In this section results will be reported from the application of the rational interpolation with optimized zeros of the denominator, as outlined in the previous section, to several typical problems. In all examples the sup–norm $\| \quad \|_\infty$ has been estimated by considering the 1000 equally spaced points

$$\widehat{x}_\ell = -\frac{5}{4} + \frac{\ell-1}{999}\frac{10}{4}, \qquad \ell = 1(1)1000,$$

on the interval $[-5/4, 5/4]$ and computing the maximal absolute value of the error at those $\widehat{x}_\ell$ lying in $[-1, 1]$; this way the points do not change with N.

To minimize $\|r - f\|_\infty$ for r as in (3) a general multi-objective nonlinear optimization algorithm, such as FFSQP in [21], cannot be utilized. The fact that the objective function is not differentiable at the interpolation nodes did not prevent the application of FFSQP in [5] because there the objective function had exactly one local maximum between adjacent nodes, permitting the minimization of these maxima between the interpolation points, thus avoiding the points of nondifferentiability. Here the objective function in general has an unknown number of maxima between adjacent nodes.

The following numerical results were obtained with two different algorithms. For small N a discrete differential correction algorithm according to [14] was used, while for larger N the simulated annealing method of [9] was applied. While the algorithm used in [5] in general finds *local* extrema, both methods used here will in principle locate a desired *global* maximum, in the first case in a systematic and guaranteed way evaluating the error not continuously but on a fine grid. The simulated annealing method cannot be guaranteed to find the global extremum but, when used for an extensive search, will produce a reasonable approximation of it.

Example 1. Since the poles are always different from the interpolation points, the denominator never vanishes at the nodes. This eliminates the risk that *unattainable points* render the problem unsolvable [19]. Let the function

$$f(x) = \sin\left(\pi(x - 0.5)\right) - \frac{16}{3\,\mathrm{Log}\,2}x(x^2 - 1)\,\mathrm{Log}\,\frac{2x + 3}{4}$$

be interpolated between five equidistant points in $[-1, 1]$. The value vector is $[1, 2, -1, 0,\ 1]$ and the point (1,1) is unattainable for the classical rational interpolant in $\mathcal{R}_{2,2}$ [6]. The latter takes the value .2 at $x = 1$, while the optimal r according to (3) with $P = 2$ has a maximal absolute error .01799. The attached poles are located at about $-2.6149 \pm 3.3794i$ and the modulus of c_1 and c_2 in (5) is about 48.93.

Example 2. As noted in the introduction, the classical rational interpolant typically exhibits annoying poles in the interval of interpolation when N is small. An example of this fact, seemingly proposed by Cordellier [20] (in a slightly different form), is the piecewise linear function taking the values

$[-\frac{1}{2}, -\frac{1}{4}, 0, 0, 0, \frac{1}{4}, 0, -\frac{1}{4}, -\frac{1}{2}]$ at the 9 equidistant points on $[-1, 1]$. Figure 1a and Figure 1b show the function and its polynomial interpolant, respectively the function and its classical rational interpolant in $\mathcal{R}_{4,4}$, while in Figure 2a and Figure 2b our interpolants with 2, respectively 4 attached poles are graphed, again together with the function. It is clear that, from an approximation point of view, our interpolant is vastly superior to the polynomial and the rational interpolants in [20, p. 281]. Table 1 lists errors and pole locations: the first column gives the number P of poles, the second the maximal error, the third and fourth one pole from each pair of conjugate complex poles and the fifth the modulus $\mathrm{abs}(c_i)$ of c_i in (5) for the corresponding poles.

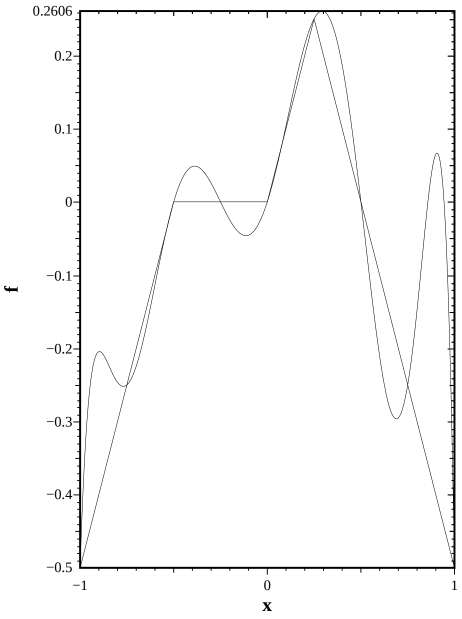

Figure 1a.

Table 1

Results for Example 2

P	max error	Re(z_i)	Im(z_i)	abs(c_i)
0	.479296			
2	.387673E–01	.151498	.302879	3.803
4	.239559E–01	–.454952	.406143	.4450
		.202325	.163394	1.148

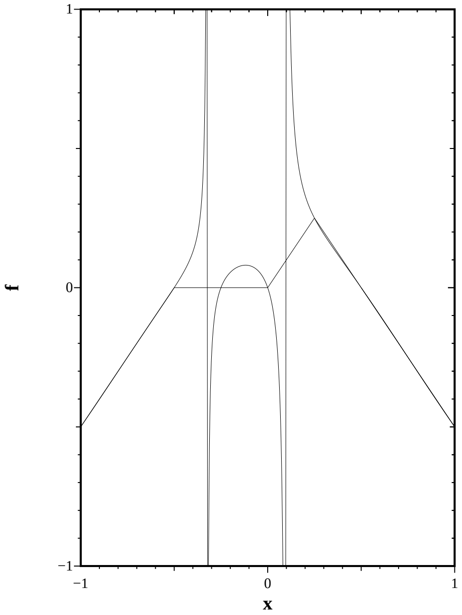

Figure 1b.

Table 2

Example 3, errors for classical rational interpolation

N	P	equidist.	Čebyšev
7	2	.1937E01	.5212E01
	4	.3826	.1416
	6	.5998E–02	.8739
15	2	.4990	.2340
	4	.2448E–01	.1014E–02
	6	.3113E–03	.1352E–04
	8	.7052E–06	.3452E–07
31	2	.2355E–01	.6927E–02
	4	.2132E–03	.1692E–06
	6	.1493E–05	.7481E–09
	8	.3160E–08	.1372E–11
63	2	.2500E–04	.9849E–08
	4	.3115E–07	.7477E–14
	6	.6136E–12	.6844E–18

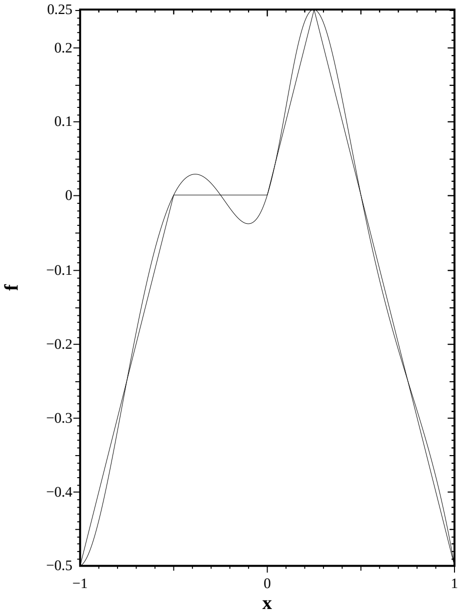

Figure 2a.

Table 3

Results for Example 3, equidistant points

N	P	max error	Re(z_i)	Im(z_i)	abs(c_i)
7	0	2.00705			
	2	.557636	−.846124	.220082	3.021
	4	.5086E–03	−1.068126	.321507E–01	.2729
			−.312498E–03	.200091	10.78
	6	.871959E–05	−1.094751	.133455E–01	1.142E–03
			−.295555E–06	.199999	14.14
			−1.099497	.610652E–01	9.286E–04
15	0	8.10977			
	2	.108660	−1.41066E–02	.112443	326.6
	4	.129434E–04	−.324645E–05	.200002	406.2
			−1.083496	.275526E–01	.1252
	6	.942704E–06	−.568190E–07	.200000	553.6
			−1.098424	.226468E–01	8.115E–04
			−1.133404	.570938E–01	2.311E–04
	8	.167838E–07	−.552288E–12	.200000	713.7
			−1.125287	.173849E–01	4.644E–07
			−1.101591	.103767E–01	4.545E–07
			−1.122610	.557111E–01	9.178E–07
31	0	2623.53			
	2	.171958E–02	−1.217508E–06	.200000	2.864E05
	4	.123776E–06	−1.099067	.228277E–01	1.095E–02
			−.990894E–10	.200000	3.576E05
	6	.419991E–07	−.131845E–09	.200000	1.131E06
			−1.100112	.223975E–01	2.515E–02
			−.9650369	1.497751	1.692E–04
	8	.204998E–09	.973004E–15	.200000	7.736E05
			−1.213014	.411931E–01	9.748E–08
			−1.147001	.224904E–03	2.780E–07
			−1.129839	.206229E–01	4.406E–07
63	0	.152604E09			
	2	.752845E–06	.112984E–15	.200000	1.431E11
	4	.485318E–10	.391884E–18	.200000	1.832E11
			−1.113237	.180890E–01	113.5
	6	.768149E–12	−.207903E–20	.200000	2.453E11
			−1.130093	.247649E–01	231.8
			−1.122707	.677363E–02	238.0

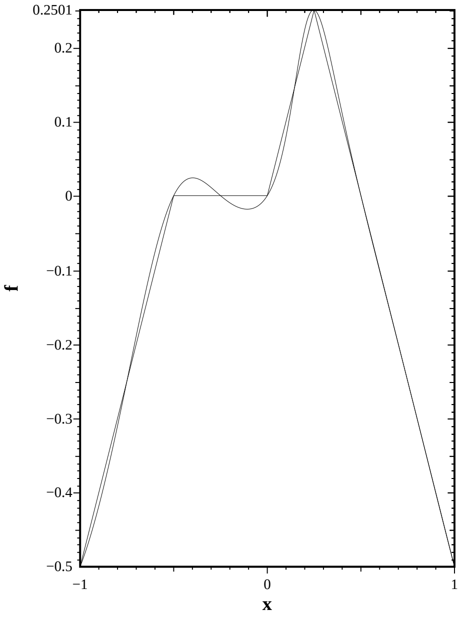

Figure 2b.

Example 3. Next we present an example where one can actually watch the distance of the poles to the singularities of the function change as N and P grow. For that purpose we consider the function $f(x) = e^{1/(x+1.2)}/(1+25x^2)$ used in [6,4]. Table 3 displays the results for equidistant interpolation points. (If N and P are not small, standard double precision calculations are not sufficient to cope with the bad condition of the problem. For that reason, and in contrast with our other results, the numbers in Table 3 were computed in quadruple precision.)

For the sake of comparison, we give in Table 2 the approximation error of the classical rational interpolant in $\mathcal{R}_{N-P,P}$ with the same number of interpolation points and poles. The barycentric form has been used as well for determining this classical interpolant by means of the kernel of the matrix (17) in [6]. The comparison shows the obvious superiority of our interpolant, especially as long as N and P are small and classical rational interpolation is unreliable.

Table 4 displays the same results for Čebyšev points of the second kind, again to be compared with those of Table 2: the results are similar.

In both tables, it is interesting to watch a pair of optimized poles approach the poles of f at $\pm.2\mathrm{i}$ as N and P increase. Note also how some other pole(s) come to lie in the vicinity of the essential singularity at $x = -1.2$, without tending toward this point, in accordance with Weierstrass' theorem on the values of a function in the neighborhood of an essential singularity [1, p. 129].

Example 4. Finally we consider a case with a large derivative in the interior of the interval of interpolation, as motivated by the introduction. With erf denoting the standard error function and for given positive ϵ the function to approximate is chosen as [11]

$$f(x) = \cos \pi x + \frac{\operatorname{erf}(\delta x)}{\operatorname{erf}(\delta)}, \qquad \delta = \sqrt{.5\epsilon}.$$

This function has values -2 at $x = -1$ and 0 at $x = 1$ and has a steep gradient near $x = 0$ for large ϵ. Figure 3 shows the graph of f for $\epsilon = 10,000$.

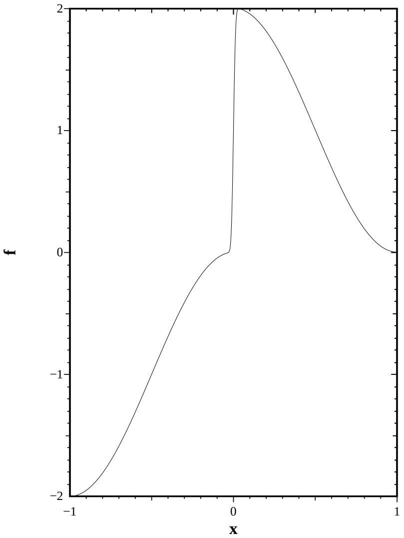

Figure 3.

For not too large values of ϵ the cases of moderate numbers P of poles could be relatively easily solved, while the problem is getting harder with increasing ϵ due to the steep gradient near $x = 0$. For example, with $\epsilon = 100$ everything works perfectly: with two pairs of poles, the error decreases exponentially with N, from $4.0 \cdot 10^{-3}$ for $N = 7$ to $4.1 \cdot 10^{-14}$ for $N = 63$, whereas the polynomial error decreases merely from $1.3 \cdot 10^{-1}$ to $2.1 \cdot 10^{-6}$. For $\epsilon = 10,000$ and Čebyšev points of the second kind, in two of the cases the algorithm used failed to produce the desired results. In all other cases, however, the numbers in Table 6 show again that the attachment of a small number of poles leads to a significant improvement of the approximation properties of the interpolant.

Note the decreasing values of $\text{abs}(c_i)$, the test for the presence of the poles, as N grows. This stems from the fact, noted above, that this quantity is a divided difference of order N: if the derivatives do not increase as fast as the corresponding factorials, divided differences become smaller as their order increases.

For small N, classical rational interpolation (Table 5) has a hard time with the latter example. Poles occur in the interpolation interval at least until $N = 15$, where our r has already decreased the error to $5.5 \cdot 10^{-3}$, a value the classical interpolant does not even reach with 64 points for as small a denominator degree.

While the generation of some of the pole coordinates in our tables took a long time, these computations were only done to show the error behavior. For practical applications more efficient algorithms may well be found.

Conclusion

In the present work we have presented rational interpolants with guaranteed interpolation, no poles in the interval of interpolation, and an error which usually decreases — and never increases — with the degree of the denominator. The error is consistently smaller than that of classical rational interpolation with the same denominator degree.

Simple comparison is not fair, however: our r has total degree $N + P$, as opposed to N for the classical rational interpolant. More importantly, it requires knowledge of the interpolated function in the entire interpolation interval, so that it is more an "interpolative approximant" than an interpolant in the classical sense. We do think, however, that it will have interesting applications.

One of them is *model reduction* in control system design. Indeed, it often happens there that a rational transfer function, $T(s)$ say, has too large numerator and/or denominator degrees m, resp. n, and that one wants to decrease these without loosing the main features of T. Several methods exist for that purpose,

Table 4

Results for Example 3, Čebyšev points of the second kind

N	P	max error	Re(z_i)	Im(z_i)	abs(c_i)
7	0	1.274540			
	2	1.041588	–.984145	.155790E–01	.2485E–01
	4	.794517E–03	–1.066884	.326360E–01	.9262E–01
			–.346010E–03	.199926	.3411
	6	.573120E–04	–1.077953	.284679E–01	.2300E–02
			–.175400E–04	.200010	.4741
			–1.142417	.132298	.2558E–03
15	0	.237659			
	2	.224479	–.849569	.118225E–01	.2066E–02
	4	.124278E–04	–.264222E–04	.199990	.1632
			–1.084673	.271153E–01	.1041E–01
	6	.565391E–06	–1.139591	.894443E–01	.1234E–04
			–.589498E–06	.200000	.2234
			–1.094256	.232891E–01	1.1587
	8	.331751E–07	–1.125766	.299623E–01	.7444E–05
			–.153199E–07	.200000	.4492
			–1.311525	.468433	.6263E–07
			–1.122292	.170493E–01	.9048E–05
31	0	.926485E–02			
	2	.179527E–03	–.241920E–02	.189619	.1062E–01
	4	.290629E–08	–1.104158	.209447E–01	.2457E–04
			–.125961E–06	.199999	.1456E–01
	6	.200699E–09	–1.171927	.882604E–01	.1141E–07
			–.528615E–08	.200000	.2091E–02
			–1.110073	.190116E–01	.3098E–06
	8	.169550E–10	–1.099230	.226391	.1820E–12
			–1.110667	.170531E–01	.4685E–08
			–1.130098	.601730E–01	.6161E–09
			–.933120E–11	.200000	.2521E–01
63	0	.159734E–04			
	2	.698346E–10	–0.554176E–06	.199998	.4071E–04
	4	.399680E–14	–.133091E–10	.200000	.5286E–04
			–1.121621	.155754E–01	.1398E–10

Table 5

Example 4, $\epsilon = 10^4$, errors for classical rational interpolation

N	P	max error
7	2	pole
	4	1.398
	6	pole
15	2	.6274
	4	pole
	6	pole
	8	pole
31	2	.3719
	4	.3154
	6	.2833
	8	.2659
63	2	.1388
	4	.1991E–01
	6	.5993
	8	.9756E–01

and their results can vary enormously [15]. The approach we suggest for replacing T with $T' \in \mathcal{R}_{m',n'}$, $m' \leq m$, $n' \leq n$, is to consider interpolation nodes $s_0, \ldots, s_{m'}$ (and corresponding interpolated values $T(s_i)$) well chosen to maintain the main features of T (e.g., its extrema). Then our method can be used to construct T' by optimally attaching n' poles to the polynomial $p \in \mathcal{P}_{m'}$ interpolating between the s_i.

Another application is the *numerical solution of two-point boundary value problems* $Lu = h$ [7]. We suggest to iteratively improve upon the classical polynomial pseudospectral (collocation) method: once approximate values u_k of the solution at the $N+1$ collocation points x_k have been found, we optimally attach poles by minimizing the residuum $\|Lr - h\|$ (with respect to P poles z_i) among all rational interpolants r of the u_k's as given in (3). The optimal z_i determine a denominator as in (3), and the set of all rational functions in $\mathcal{R}_{N,P}$ interpolating between the x_k's and sharing this common denominator form a linear space. The second step of our iteration procedure consists in solving the original problem $Lu = h$ in this space by the linear rational collocation method [3]. We then simply repeat the two steps described above until convergence, see [7] for details. Numerical tests are encouraging.

A final remark: the alert reader will wonder why we did not simply minimize

Table 6

Results for Example 4, $\epsilon = 10^4$, Čebyšev points of the second kind

N	P	max error	Re(z_i)	Im(z_i)	abs(c_i)
7	0	.860929			
	2	.585487	.498187E−01	.855217E−01	.9510E−01
	4	.250594	.963782	.195789	.1388
			.173890E−02	−.519644E−01	.3671E−01
	6	.136934	.381364	−.503092	.4074
			−1.490736	.527043	.1002
			.218230E−02	−.363523E−01	.1622E−01
15	0	.731061			
	2	.152567	.445251E−07	.386686E−01	.1594E−01
	4	.129811E−01	.691377E−05	−.226144E−01	.1851E−03
			.714004E−04	−.148341	.2528E−02
	6	.550262E−02	.178129E−01	.334510	.1433E−03
			.199111E−02	−.105384	.1369E−03
			.180070E−04	.209114E−01	.2179E−04
31	0	.527525			
	2	.347874E−01	−.303433E−12	−.251649E−01	.2897E−03
	4	.609649E−02	−.899892E−12	.994387E−01	.1162E−03
			−.279530E−12	.207341E−01	.2197E−04
	6				
63	0	.269966			
	2	.612221E−02	.378870E−09	−.208431E−01	.3708E−09
	4				
	6	.808776E−03	.628162E−02	−.190003E−01	.3167E−10
			−.118652E−08	.694756E−01	.1117E−09
			−.628161E−02	.190003E−01	.3167E−10
127	0	.102178			
	2	.2822739E−02	−.263671E−03	.217792E−01	.2285E−15
	4	.584158E−03	−.674335E−02	−.204741E−01	.1388E−14
			.674335E−02	.204741E−01	.1110E−14
	6	.143965E−04	.156860E−01	−.249369E−01	.1221E−14
			−.156860E−01	.249370E−01	.7772E−15
			−.106757E−07	.263960E−01	.2224E−15

the approximation error with respect to all of the w_k's in (1) — after all, as mentioned in §2, every rational interpolant in $\mathcal{R}_{N,N}$ can be written as such a barycentric expression. In fact, this is the way we started, but we encountered difficulties, both on the theoretical side (existence of an optimum, of an alternating sequence, etc.) as on the practical side (too many parameters to optimize). That led us to the present compromise of merely optimizing low degree denominators, which gives satisfactory results in many cases, as demonstrated in our experiments.

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