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Optimal Input Signal Design in Data-Centric System Identification

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Abstract

System identification is the science of modelling the behavior of a dynamical system based on input-output data. Input signal design is a critical aspect of system identification since it directly affects the model quality. Highly interactive multiple input multiple output (MIMO) systems present challenges in implementing methodologies like Model on Demand (MoD) that require a good output state space distribution.

A powerful optimization-based framework for input signal design is presented that achieves very good distribution in the output state space and, in addition, makes the input signal "plant friendly". This new framework was tested in case studies of increasing difficulty. For each case study, an optimization problem was formulated using this new framework and solved using the NLP solver KNITRO and the modelling language AMPL.

The results show the excellent directionality information obtained using this approach for both linear and non-linear systems and the great value of this framework for MoD. We present graphical results for the most difficult case study and compare to the best results obtained with crest factor minimization.

1. Introduction to System Identification

Dynamical Systems occur all around us. In a dynamical system variables of various types interact with each other and produce signals of interest to us called outputs.

The process of constructing mathematical models of a dynamical system based on observed input and output data is called system identification. The variables of a dynamical system in which one is interested are called output variables. The variables which affect the internal state (and thereby the output variables) of a dynamical system are called the input variables. Those input variables that we can control are called manipulated variables, while those that cannot be controlled are called disturbance variables.

1.1. Steps Involved in System Identification

We consider an introductory example [7] to bring out the various aspects of system identification. Suppose a dynamical system has one input and one output, $u(t)$ and $y(t)$ respectively. We observe the system from time $t = 1, \cdots, N$ and collect data $\{u(1), y(1), u(2), y(2), \cdots, u(N), y(N)\}$. We postulate either empirically or by using physical insight that $y(t)$ is related to $u(t)$ by the difference equation

$$y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) = b_1 u(t-1) + \cdots + b_m u(t-m) \quad (1)$$

or in matrix form

$$\hat{y} = \phi^T(t)\theta \quad (2)$$

where $$\phi(t) = [-y(t-1), \ldots, -y(t-n), u(t-1), \ldots, u(t-m)]^T \quad (3)$$

$$\theta = [a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_m]^T \quad (4)$$

$$\hat{y}(t) = \text{“calculated” } y(t) \text{ from the model} \quad (5)$$

If the system is at steady state at $t \le 0$, then $y(0), y(-1), \ldots, y(1-n) = 0$ and $u(0), u(-1), \ldots, u(1-m) = 0$. We can choose the criterion to evaluate or fit our model to the collected data. In this simple example, we use the least squares approach. To choose $a_1, a_2, \ldots, b_1, b_2, \ldots, b_m$ or θ in a “best” possible way, we define the objective function

$$V(\theta) = \frac{1}{N}\sum_{t=1}^{N}(y(t) - \hat{y}(t))^2 = \frac{1}{N}\sum_{t=1}^{N}(y(t) - \phi^T(t)\theta)^2 \quad (6)$$

Next we use optimization to minimize V. Let

$$\hat{\theta} = \arg\min_{\theta} V(\theta) \quad (7)$$

The optimization procedure itself might be quite complicated in more difficult examples. In this particular example it is straightforward to minimize V. We set the gradient of $V, \nabla V$ to zero.

$$\frac{\partial V}{\partial \theta_i} = \frac{1}{N}\sum_{t=1}^{N} 2(y(t) - \phi^T(t)\theta(-\phi_i(t)) = 0 \quad (8)$$

$$i = 1, 2, \ldots, n+m$$

Here θ_i denotes the i-th component of θ and $\phi_i(t)$ denotes the i-th component of $\phi(t)$. Thus,

$$\sum_{t=1}^{N} \phi_i(t)y(t) = \sum_{t=1}^{N} \phi_i(t)\phi^T(t)\theta \tag{9}$$

$$i = 1, 2, \ldots, n + m$$

Combining all equations in a matrix representation,

$$\sum_{t=1}^{N} \phi(t)y(t) = \sum_{t=1}^{N} \phi(t)\phi^T(t)\theta \tag{10}$$

Solving for $\hat{\theta} = \text{argming}_\theta V(\theta)$, we get:

$$\hat{\theta} = \left\{ \sum_{t=1}^{N} \phi(t)\phi^T(t) \right\}^{-1} \left(\sum_{t=1}^{N} \phi(t)y(t) \right) \tag{11}$$

Model structures, such as the one used above which is linear in θ are known as linear regressions. Components of $\phi(t)$ are called regressors. In the model used above, values of $y(t)$ depend on previous values of y, such models are called auto-regressions. Thus the model used above is called the ARX model (autoregressions with extra inputs).

1.2. Why is Input Design Important?

Suppose $n = 0$ in the above model. Then from (1)

$$y(t) = b_1 u(t-1) + \cdots + b_m u(t-m) \tag{12}$$

As we shall see later such an equation can always be used to describe a linear time invariant system for large enough m [see equation (67)]. Suppose this is the model that we use to describe a particular dynamical system. Suppose further that the system is actually described by the equation

$$y(t) = b'_1 u(t-1) + \cdots + b'_m u(t-m) + e(t) \tag{13}$$

Here $e(t)$ represents a disturbance signal. Suppose $e(t)$ is a "white noise". Then it can be described by a series of independent random variables with mean 0 and variance λ. Writing (13) in a matrix form

$$y(t) = \phi^T(t)\theta_0 + e(t) \tag{14}$$

where
$$\phi(t) = [u(t-1), \ldots, u(t-m)]^T \tag{15}$$

and
$$\theta_0 = [b'_1, b'_2, \ldots, b'_m]^T$$

Replacing $y(t)$ in (11) by (14) we get

$$\hat{\theta} = R(N)^{-1}\left[\sum_{t=1}^{N}\phi(t)\theta_0 + \sum_{t=1}^{N}\phi(t)e(t)\right] \tag{16}$$

Here

$$R(N) = \sum_{t=1}^{N}\phi(t)\phi^T(t) \tag{17}$$

Denote the difference between predicted and actual θ by $\tilde{\theta}$. So

$$\tilde{\theta} = \hat{\theta} - \theta_0 \tag{18}$$

From equations (16) and (17)

$$\hat{\theta} = \theta_0 + R(N)^{-1}\sum_{t=1}^{N}\phi(t)e(t) \tag{19}$$

Thus

$$\tilde{\theta} = \hat{\theta} - \theta_0 = R(N)^{-1}\sum_{t=1}^{N}\phi(t)e(t) \tag{20}$$

If E denotes mathematical expectation then

$$E(\tilde{\theta}) = E(\hat{\theta} - \theta_0) = R(N)^{-1}\sum_{t=1}^{N}\phi(t)Ee(t) = 0 \tag{21}$$

as $e(t)$ has a mean of zero. The estimate $\hat{\theta}$ is thus unbiased.

Let P_N denote the expectation of $\tilde{\theta}\tilde{\theta}^T$, the covariance matrix of the parameter error. Then

$$\begin{aligned}
P_N = E(\tilde{\theta}\tilde{\theta}^T) &= ER(N)^{-1}\sum_{t=1}^{N}\phi(t)e(t)\left\{R(N)^{-1}\sum_{s=1}^{N}\phi(s)e(s)\right\}^T \\
&= ER(N)^{-1}\sum_{t=1}^{N}\phi(t)e(t)\sum_{s=1}^{N}\phi^T(s)e(s)R(N)^{-1} \\
&= ER(N)^{-1}\sum_{t=1}^{N}\sum_{s=1}^{N}\phi(t)\phi^T(s)e(t)e(s)R(N)^{-1} \\
&= R(N)^{-1}\sum_{t=1}^{N}\sum_{s=1}^{N}\phi(t)\phi^T(s)Ee(t)e(s)R(N)^{-1}
\end{aligned}$$

Since $e(t)$ is a sequence of independent variables

$$E(e(t)e(s)) = \lambda\delta(t-s) \text{ where } \delta(0) = 1 \text{ and } \delta(x) = 0 \text{ if } x \neq 0. \tag{22}$$

Thus,

$$P_N = \lambda R(N)^{-1} \tag{23}$$

This example serves to illustrate various steps in the system identification procedure.

1. **Selection of Data Set/Experimental Design**
   The input-output data can be recorded during a specially designed experiment where the user chooses specific inputs and measures the corresponding outputs.
   The choice of inputs is not arbitrary. To see this, note that for the example presented above
   $$R_{ij}(N) = \sum_{t=1}^{N} u(t-i)u(t-j) \tag{24}$$
   The goal in choosing $u(t)$ for the experiment should be to choose it in such a way that
   (a) $R(N)$ is invertible and
   (b) $||R(N)^{-1}||$ is minimized
   This is desirable since then P_N will also be "small" and $\hat{\theta}$ (predicted) will agree well with θ_0(actual).
2. **Selection of Model Set**
   This is the most important step in system identification process. A model set is a set of models with similar structure.
   In some cases, it may be possible to use the principles of conservation of mass, energy or transport principles to derive models with various adjustable parameters. In other cases, one may represent the system using standard linear models (linear difference equations) without using any physical insight (the black box models).
3. **Selection of One Particular Model from the Model Set**
   One then selects a particular model from the model set based on some objective function. For example in (6) we used the least square parameter error criterion. One may decide to use some other choice criterion based on one's requirements.
4. **Model Validation**
   After completing steps 1, 2 and 3 we have a working model which is the best possible with regard to a particular criterion. The model is then tested to see if it is good enough using various tests. Such tests are called model validation.

1.3. Dynamical Systems and Transfer Functions

Our goal in this section is to describe the various time-invariant lumped parameter models used to describe dynamical systems.

Suppose the internal state of a dynamical system is described by a vector of internal state variables $x \in \mathbf{R}^n$. The input to the dynamical system is a vector $u \in \mathbf{R}^m$ that directly affects the state vector x and can be controlled by the user. Another type of input to the dynamical system is the so called disturbance. This is described by the vector $d \in \mathbf{R}^q$ which cannot be controlled by the user. The output variables y are the input or state variables that are of interest to us. In general, such a dynamical system can be described by:

$$\frac{dx}{dt} = f(x, u, d), \text{ where } f \in \mathbf{R}^n \tag{25}$$

$$y = g(x, u, d), \text{ where } g \in \mathbf{R}^p \tag{26}$$

A dynamical system is said to be at an equilibrium or steady state when the vector x describing its internal state remains constant. This happens when

$$\frac{dx}{dt} = 0, \text{ or when } f(x, u, d) = 0 \tag{27}$$

Obviously, (27) can have many solutions. So a dynamical system can have many steady states. It is difficult to consider the full nonlinear version of the dynamical system described by (25) and (26). Thus it is a common practice to "linearize" the dynamical system about a particular equilibrium point or steady state. Suppose $(\overline{x}, \overline{u}, \overline{d})$ is a steady state of the dynamical system. Then

$$f(\overline{x}, \overline{u}, \overline{d}) = 0 \tag{28}$$

We want to replace $f(x, u, d)$ and $g(x, u, d)$ by their linear versions using a Taylor series expansion about $(\overline{x}, \overline{u}, \overline{d})$.

$$\begin{aligned} f(x,u,d) \approx {} & f(\overline{x}, \overline{u}, \overline{d}) + J_x^f(x - \overline{x}) + J_u^f(u - \overline{u}) + J_d^f(d - \overline{d}) + O(||x - \overline{x}||^2 \\ & + ||u - \overline{u}||^2 + ||d - \overline{d}||^2) \end{aligned} \tag{29}$$

$$\begin{aligned} g(x,u,d) \approx {} & g(\overline{x}, \overline{u}, \overline{d}) + J_x^g(x - \overline{x}) + J_u^g(u - \overline{u}) + J_d^g(d - \overline{d}) + O(||x - \overline{x}||^2 \\ & + ||u - \overline{u}||^2 + ||d - \overline{d}||^2) \end{aligned} \tag{30}$$

Here,

$$J_x^f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{(\overline{x}, \overline{u}, \overline{d})} \in \mathbf{R}^n \times \mathbf{R}^n \tag{31}$$

$$J_u^f = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{pmatrix}_{(\overline{x}, \overline{u}, \overline{d})} \in \mathbf{R}^n \times \mathbf{R}^m \tag{32}$$

$$J_d^f = \begin{pmatrix} \frac{\partial f_1}{\partial d_1} & \frac{\partial f_1}{\partial d_2} & \cdots & \frac{\partial f_1}{\partial d_q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial d_1} & \frac{\partial f_n}{\partial d_2} & \cdots & \frac{\partial f_n}{\partial d_q} \end{pmatrix}_{(\overline{x},\overline{u},\overline{d})} \in \mathbf{R}^n \times \mathbf{R}^q \tag{33}$$

$$J_x^g = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \frac{\partial g_p}{\partial x_2} & \cdots & \frac{\partial g_p}{\partial x_n} \end{pmatrix}_{(\overline{x},\overline{u},\overline{d})} \in \mathbf{R}^p \times \mathbf{R}^n \tag{34}$$

$$J_u^g = \begin{pmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u_1} & \frac{\partial g_p}{\partial u_2} & \cdots & \frac{\partial g_p}{\partial u_m} \end{pmatrix}_{(\overline{x},\overline{u},\overline{d})} \in \mathbf{R}^p \times \mathbf{R}^m \tag{35}$$

$$J_d^g = \begin{pmatrix} \frac{\partial g_1}{\partial d_1} & \frac{\partial g_1}{\partial d_2} & \cdots & \frac{\partial g_1}{\partial d_q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial d_1} & \frac{\partial g_p}{\partial d_2} & \cdots & \frac{\partial g_p}{\partial d_q} \end{pmatrix}_{(\overline{x},\overline{u},\overline{d})} \in \mathbf{R}^p \times \mathbf{R}^q \tag{36}$$

Since from (28)$f(\overline{x},\overline{u},\overline{d}) = 0$, (29) becomes,

$$f(x,u,d) \approx J_x^f(x-\overline{x}) + J_u^f(u-\overline{u}) + J_d^f(d-\overline{d}) \tag{37}$$

If $\overline{y}$ is the steady state value of the output then obviously

$$\overline{y} = g(\overline{x},\overline{u},\overline{d}) \tag{38}$$

With the linear approximations for f and g we get a linearized system about $(\overline{x},\overline{u},\overline{d})$.

$$\frac{dx}{dt} = \frac{d}{dt}(x-\overline{x}) = J_x^f(x-\overline{x}) + J_u^f(u-\overline{u}) + J_d^f(d-\overline{d}) \tag{39}$$

Also from (38) and (30), we get

$$y-\overline{y} = g(x,u,d) - g(\overline{x},\overline{u},\overline{d}) = J_x^g(x-\overline{x}) + J_u^g(u-\overline{u}) + J_d^g(d-\overline{d}) \tag{40}$$

We now introduce "deviation" variables which measure the deviation from steady state.

$$\begin{aligned} x' &= x-\overline{x} \\ u' &= u-\overline{u} \\ d' &= d-\overline{d} \\ y' &= y-\overline{y} \end{aligned}$$

Thus equations (39) and (40) become

$$\frac{dx'}{dt} = J_x^f x' + J_u^f u' + J_d^f d' \tag{41}$$

$$y' = J_x^g x' + J_u^g u' + J_d^f d' \tag{42}$$

For notational simplicity we drop primes from (41) and (42) and replace J_x^f by A, J_u^f, by B, J_x^g by C, J_u^g by D_u, J_d^f by E and J_d^g by F. We thus get:

$$\frac{dx}{dt} = Ax + Bu + Ed \tag{43}$$

$$y = Cx + D_u u + Fd \tag{44}$$

Note that in (43) and (44), x, u, d and y are understood to be deviation variables, i.e. deviations from the steady state. It will be convenient to convert the differential-algebraic equations (43) and (44) into an algebraic system. This is achieved by taking the Laplace transform of these equations. The Laplace transform of a function $f(t)$ is defined by

$$F(s) = L\{f(t)\} \int_0^\infty e^{-st} f(t) dt \tag{45}$$

Now

$$L\left\{\frac{dx}{dt}\right\} = \int_0^\infty e^{-st} \frac{dx}{dt} dt$$

Integrating by parts

$$\begin{aligned} L\left\{\frac{dx}{dt}\right\} &= (e^{-st}x(t))_0^\infty - \int_0^\infty (-s)e^{-st}x(t)dt \\ &= s\int_0^\infty e^{-st}x(t)dt - x(0) \end{aligned} \tag{46}$$

If we assume that the system was at a steady state at $t = 0$ then since x is a deviation variable $x(0) = 0$ and so

$$L\left\{\frac{dx}{dt}\right\} = sX(s) \tag{47}$$

where

$$X(s) = \int_0^\infty e^{-st} x(t) dt \tag{48}$$

is the laplace transform of $x(t)$. Taking the laplace transform of equations (43) and (44)

$$L\left\{\frac{dx}{dt}\right\} = L\{Ax + Bu + Ed\} \tag{49}$$

$$L\{y\} = L\{Cx + D_u u + Fd\} \tag{50}$$

Using (47) and the fact that L is a linear operator, we get

$$sX(s) = AX(s) + BU(s) + ED(s) \tag{51}$$

$$Y(s) = CX(s) + D_uU(s) + FD(s) \tag{52}$$

From equation (51), we get

$$X(s) = (sI_n - A)^{-1}\{BU(s) + ED(s)\} \tag{53}$$

Here, I_n represents the $n \times n$ identity matrix. Substituting equation (53) into equation (52), we get

$$Y(s) = P_u(s)U(s) + P_d(s)D(s) \tag{54}$$

where $$P_u(s) = C(sI_n - A)^{-1}B + D_u \tag{55}$$

and $$P_d(s) = C(sI_n - A)^{-1}E + F \tag{56}$$

Equation (54) gives a straightforward way to estimate $y(t)$ based on information about $u(t)$ and $d(t)$.

1. Compute $U(s)$ and $D(s)$ from $u(t)$ and estimates of $d(t)$
2. Compute $Y(s)$ using (54) and
3. Compute $y(t)$ by taking the inverse laplace transform of $Y(s)$

1.4. Models of Time Invariant Linear Systems

The purpose of this and following sections is to introduce some terminology commonly used in system identification literature. This will serve as background material which will be useful later.

As seen above, a non-linear model is often linearized about a steady state to study the properties of a dynamical system in the vicinity of that steady state. A system is called time-invariant if its dynamical response to a particular input does not depend on time. Suppose such a time invariant linear system relates the input $u(t)$ to the output $y(t)$ of the system by the transfer function $G_c(s)$

$$Y(s) = G_c(s)U(s) \tag{57}$$

where $U(s)$ and $Y(s)$ are the laplace transforms of the input and output respectively and $G_c(s)$ is the transfer function of the system. Taking the inverse laplace transform on both sides of equation (57) and using the deconvolution theorem, we get

$$L^{-1}\{Y(s)\} = L^{-1}\{G_c(s)U(s)\} \tag{58}$$

$$y(t) = \int_0^t g_c(\tau)u(t-\tau)d\tau \tag{59}$$

Here, $$g_c(t) = L^{-1}\{G_c(s)\} \tag{60}$$

is the impulse response of the system. Knowing the impulse response $g_c(t)$ for the system allows us to calculate the output for any given input. The impulse response therefore describes the system completely.

It is clear that the long term behavior of the system is given by replacing the upper limit of integration by ∞ in equation (58). Thus

$$y(t) = \int_0^\infty g_c(\tau)u(t-\tau)d\tau \tag{61}$$

1.5. Discrete Time Formulation

In actual practice, information about the input is not available as a functional description. Input values are known at certain equally spaced (usually) instants of time. We will assume that the output $y(t)$ is observed at equally spaced intervals of time called sampling instants

$$t_k = kT, k = 1, 2, \ldots \tag{62}$$

The interval T is referred to as the sampling interval. It is a common practice to keep the input signal constant between sampling instants. We thus have

$$u(t) = u_k, kT \leq t < (k+1)T \tag{63}$$

Replacing the continuous time t by the discrete time kT in equation (61), we get

$$y(kT) = \int_0^\infty g_c(\tau)u(kT-\tau)d\tau \tag{64}$$

Note that $u(kT+\tau) = u_{k-1}$ when $(l-1)T < \tau \leq lT$. Therefore

$$\begin{aligned} y(kT) &= \int_0^\infty g_c(\tau)u(kT-\tau)d\tau \\ &= \sum_{l=1}^{\infty} \int_{\tau=(l-1)T}^{lT} g_c(\tau)u(kT-\tau)d\tau \\ &= \sum_{l=1}^{\infty} \left\{ \int_{\tau=(l-1)T}^{lT} g_c(\tau)d\tau \right\} u_{k-l} \\ y(kT) &= \sum_{l=1}^{\infty} g_T(l)u_{k-l} \end{aligned} \tag{65}$$

where

$$g_T(l) = \int_{(l-1)T}^{lT} g_c(\tau)d\tau \tag{66}$$

The above equation describes a so called sampled data system. Even if the input is not constant between sampling intervals, the above equation is still a reasonable approximation provided the input $u(t)$ does not change too much during a sampling interval.

For ease of notation and in line with the standard terminology [7] we assume without loss of generality that $T = 1$ unit and use t to enumerate the sampling instants. We also replace q_T by g. Equation (65) thus becomes

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k), \quad t = 0, 1, 2, \ldots \tag{67}$$

1.6. Disturbances

According to equation (67) the output can be calculated exactly once the input is known. This is unrealistic because of the presence of "disturbances" which also affect the system. Examples of disturbances could be uncontrolled inputs and measurement noise. We assume that such effect can be lumped into an additive term $v(t)$ at the output. Thus

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k) + v(t), \quad t = 0, 1, 2, \ldots \tag{68}$$

The disturbance term $v(t)$ in the above equation is not known beforehand. It is commonly modelled as follows:

$$v(t) = \sum_{k=0}^{\infty} h(k)e(t-k), \quad t = 0, 1, 2, \ldots \tag{69}$$

Here, $e(t)$ is the so called "white noise". $e(t)$ is a sequence of independent, identically distributed random variables with a certain probability density function. It is commonly assumed that the sequence $e(t)$ has zero mean and variance λ and that $h(0) = 1$ for normalization purposes. With this description, the expected value of $v(t)$ is

$$Ev(t) = \sum_{k=0}^{\infty} h(k)Ee(t-k) = 0 \tag{70}$$

and the covariance of $v(t)$ is

$$\begin{aligned} Ev(t)v(t-\tau) &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} h(k)h(m)Ee(t-k)e(t-\tau-m) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} h(k)h(m)\delta(k-\tau-m)\lambda \\ Ev(t)v(t-\tau) &= \lambda \sum_{k=0}^{\infty} h(k)h(k-\tau) \end{aligned} \tag{71}$$

Here $\delta(x)$ is as defined in (22). We note that the covariance is independent of t and define the covariance function of $v(t)$ as follows:

$$R_v(\tau) = \lambda \sum_{k=0}^{\infty} h(k)h(k-\tau) \tag{72}$$

Since the expected value $Ev(t)$ and the covariance function $Rv(\tau)$ are independent of t, the process $v(t)$ is called a stationary process.

1.7. Discrete Time Transfer Functions

We introduce the so called forward and backward shift operators q and q^{-1} respectively. They are defined as follows:

$$qu(t) = u(t+1) \tag{73}$$

and

$$q^{-1}u(t) = u(t-1) \tag{74}$$

Now we can rewrite equation (67) as follows:

$$\begin{aligned} y(t) &= \sum_{k=1}^{\infty} g(k)u(t-k), \quad t = 0,1,2,\dots \\ &= \sum_{k=1}^{\infty} g(k)(q^{-k})u(t), \quad t = 0,1,2,\dots \\ &= \left[\sum_{k=1}^{\infty} g(k)g^{-k}\right] u(t), \quad t = 0,1,2,\dots \\ y(t) &= G(q)u(t) \end{aligned} \tag{75}$$

where

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k} \tag{76}$$

is the so called transfer operator or the transfer function of the linear system (67).

In a similar way we can write

$$v(t) = H(q)e(t), \quad t = 0,1,2,\dots \tag{77}$$

where

$$H(q) = \sum_{k=0}^{\infty} h(k)q^{-k} \tag{78}$$

Equation (68) can thus be written in transfer function notation as

$$y(t) = G(q)u(t) + H(q)e(t), \quad t = 0,1,2,\dots \tag{79}$$

where $e(t)$ is a sequence of independent random variables with zero mean and variance λ. The function $G(q)$ is also sometimes referred to as a "filter". A filter $G(q)$ is said to be stable if

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k}, \quad \sum_{k=1}^{\infty} |g(k)| < \infty \tag{80}$$

and strictly stable if

$$G(q) = \sum_{k=1}^{\infty} g(k) q^{-k}, \quad \sum_{k=1}^{\infty} k|g(k)| < \infty \tag{81}$$

A filter $H(q)$ is said to be monic if $h(0) = 1$.

1.8. Frequency Function

Suppose the system (67) is subjected to a sinusoidal input

$$u(t) = \cos(\omega t) = \text{ Re } e^{i\omega t} \quad t = 0, 1, 2, \ldots \tag{82}$$

The output is given by

$$\begin{aligned} y(t) &= \sum_{k=1}^{t} g(k) \text{ Re } e^{i\omega(t-k)} \\ &= \sum_{k=1}^{t} g(k) \text{ Re } e^{i\omega(t-k)} - \sum_{k=t+1}^{\infty} g(k) \text{ Re } e^{i\omega(t-k)} \\ &= \text{Re } \sum_{k=1}^{\infty} g(k) e^{i\omega(t-k)} - \text{Re } \sum_{k=t+1}^{\infty} g(k) e^{i\omega(t-k)} \\ &= \text{Re } \left\{ e^{i\omega t} \sum_{k=1}^{\infty} g(k) e^{-i\omega k} \right\} - \text{ Re } \left\{ e^{i\omega t} \sum_{k=t+1}^{\infty} g(k) e^{-i\omega k} \right\} \\ &= \text{Re } \{ e^{i\omega t} G(e^{i\omega}) \} - \text{Re } \left\{ e^{i\omega t} \sum_{k=t+1}^{\infty} g(k) e^{-i\omega k} \right\} \\ y(t) &= |G(e^{i\omega})| \cos(\omega t + \phi) - \text{Re} \left\{ e^{i\omega t} \sum_{k=t+1}^{\infty} g(k) e^{-i\omega k} \right\} \end{aligned} \tag{83}$$

where

$$\phi = \arg G(e^{i\omega}) \tag{84}$$

Now

$$\text{Re } \left\{ e^{i\omega t} \sum_{k=t+1}^{\infty} g(k) e^{-i\omega k} \right\} \le \left| e^{i\omega t} \sum_{k=t+1}^{\infty} g(k) e^{-i\omega k} \right| \le \sum_{k=t+1}^{\infty} |g(k)| \tag{85}$$

But

$$\sum_{k=t+1}^{\infty} |g(k)| < \infty \tag{86}$$

if the filter $G(q)$ is stable. Further

$$\sum_{k=t+1}^{\infty} |g(k)| \to 0 \quad \text{as} \quad t \to \infty \tag{87}$$

Therefore, the long term response ($t \to \infty$) of a stable system is given by

$$y(t) = |G(e^{i\omega})| \cos(\omega t + \phi) \tag{88}$$

The above equation tells us that the output is also a sinusoid with an amplitude magnified by a factor of $G(e^{i\omega})$ and a phase shift of arg $|G(e^{i\omega})|$. Therefore the transfer function $G(g)$ evaluated at the point $q = e^{i\omega}$ gives us complete information about the steady state output from the process to a sinusoidal input of frequency ω. Note that $G(e^{i\omega})$ is periodic in ω with a period of $|2\pi|$.

$$G(e^{i\omega}) = G(e^{i(\omega+2\pi)}) \tag{89}$$

The complex valued function

$$G(e^{i\omega}), \quad -\pi \leq \omega < \pi \tag{90}$$

is called the frequency function of the system. It is often documented as a plot of log $\left(|G(e^{i\omega})|\right)$ and arg $G(e^{i\omega})$ versus $\log(\omega)$, the so called Bode plot. The plot of Re $G(e^{i\omega})$ versus Im $G(e^{i\omega})$ in the complex plane is referred to as the Nyquist plot.

1.9. Discrete Fourier Transform (DFT) on Finite Intervals

Discrete fourier transform enables us to capture the frequency content in discrete time signals. Consider a signal $u(t)t = 1, 2, \ldots, N$ where N is even. The discrete fourier transform of $u(t)$ is defined by [7]

$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t) e^{-i\omega t} \text{ where } \omega = \frac{2\pi k}{N}, \quad k = 1, \ldots, N \tag{91}$$

$u(t)$ can then be represented by the inverse DFT as follows:

$$u(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} U_N\left(\frac{2\pi k}{N}\right) e^{\frac{i2\pi kt}{N}} \tag{92}$$

$$\text{Note that} \qquad U_N(\omega + 2\pi) = U_N(\omega) \quad \text{and} \quad U_N(-\omega) = \overline{U_N(\omega)} \tag{93}$$

Now from equation (92)

$$u(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{\frac{N}{2}} U_N\left(\frac{2\pi k}{N}\right) e^{\frac{i2\pi kt}{N}} + \frac{1}{\sqrt{N}} \sum_{k=\frac{N}{2}+1}^{N} U_N\left(\frac{2\pi k}{N}\right) e^{\frac{i2\pi kt}{N}} \tag{94}$$

Now for $k = \frac{N}{2} + m$ where $m = 1, \ldots, \frac{N}{2}$

$$U_N\left(\frac{2\pi k}{N}\right) = U_N\left(\frac{2\pi m}{N} + \pi\right) = U_N\left(\frac{2\pi m}{N} - \pi\right) = U_N\left[\frac{2\pi}{N}\left(m - \frac{N}{2}\right)\right] \tag{95}$$

In view of equation (95) equation (92) can be written as

$$u(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{\frac{N}{2}} U_N\left(\frac{2\pi k}{N}\right) e^{\frac{i2\pi kt}{N}} + \frac{1}{\sqrt{N}} \sum_{k=-\frac{N}{2}+1}^{0} U_N\left(\frac{2\pi k}{N}\right) e^{\frac{i2\pi kt}{N}} \tag{96}$$

Therefore,

$$u(t) = \frac{1}{\sqrt{N}} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} U_N\left(\frac{2\pi k}{N}\right) e^{\frac{i2\pi kt}{N}} \tag{97}$$

1.10. Signal Spectrum

The discrete fourier transform defines the frequency content of a signal of finite length. To extend the concept to signals over the interval $[1, \infty)$ it is natural to consider

$$\lim_{N\to\infty} |S_N(\omega)|^2$$

It can be shown that this limit fails to exist for many signals of practical interest.

Signals like $y(t)$ in equation (79) contain a deterministic component $G(q)u(t)$ and a component $H(q)e(t)$ which is described in terms of random variables. Such a process is called a stochastic process. To develop a common framework for deterministic and stochastic signals it is useful to define a "quasi-stationary" process as described in [7]

A signal $\{\omega(t)\}$ is said to be quasi-stationary if the following conditions hold:

1. $Ew(t) = m_w(t)$ where $|m_w(t)| \leq C, \quad \forall t$
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Ew(t)w(t-\tau) = R_w(\tau), \quad \forall \tau.$

We shall define the covariance function of a quasi-stationary signal $\{s(t)\}$ as follows:

$$R_s(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Es(t)s(t-\tau) \tag{98}$$

Two signals $\{s(t)\}$ and $\{w(t)\}$ are said to be jointly quasi-stationary if they both are quasi-stationary and the cross-covariance function $R_{sw}(\tau)$ exists

$$R_{sw}(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Es(t)w(t-\tau) \tag{99}$$

Signals $\{s(t)\}$ and $\{w(t)\}$ are said to be uncorrelated if their cross-covariance function is zero. The spectrum of a quasi stationary signal $\{s(t)\}$ is defined as

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)e^{-i\omega\tau} \tag{100}$$

It is easy to prove that $\Phi_s(\omega) \geq 0 \ \forall\omega$ and that $\Phi_s(\omega)$ is real. Similarly the cross spectrum between signals $\{s(t)\}$ and $\{u(t)\}$ is defined as

$$\Phi_{su}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{su}(\tau)e^{-i\omega\tau} \tag{101}$$

One can think of $\Phi_s(\omega)$ as the fourier transform of $R_s(\tau)$. Therefore $R_s(0)$ is given by

$$R_s(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_s(\omega)d\omega \tag{102}$$

Note that

$$R_s(0) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} Es^2(t) \tag{103}$$

Introduce the notation

$$\overline{E}f(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} Ef(t) \tag{104}$$

Then from equations (102), (103) and (104), we get the so-called Parseval's relationship

$$\overline{E}s^2(t) = R_s(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_s(\omega)d\omega \tag{105}$$

Suppose a process $\{v(t)\}$ is described by

$$v(t) = H(q)e(t) = \sum_{k=0}^{\infty} h(k)e(t-k) \tag{106}$$

where $\{e(t)\}$ is a sequence of independent random variables of mean zero and variance λ. It was shown before (see equation 71) that the covariance of $v(t)$ is independent of t. Hence

$$R_v(\tau) = \overline{E}v(t)v(t-\tau) = Ev(t)v(t-\tau) = \lambda\sum_{k=0}^{\infty} h(k)h(k-\tau) \tag{107}$$

Using the definition of spectrum, we get

$$\Phi_v(\omega) = \sum_{\tau=-\infty}^{\infty} R_v(\tau)e^{-i\tau\omega} \tag{108}$$

Substituting (107) in (108), we get

$$\Phi_v(\omega) = \sum_{\tau=-\infty}^{\infty} \left\{\lambda\sum_{k=0}^{\infty} h(k)h(k-\tau)\right\}e^{-i\tau\omega} \tag{109}$$

Substitute $(k - \tau) = s$

$$\Phi_v(\omega) = \lambda \sum_{s=-\infty}^{\infty} \sum_{k=0}^{\infty} h(k)h(s)e^{-ik\omega}e^{i\omega s} \tag{110}$$

Noting that $h(r) = 0$ if $r < 0$ and rearranging, we get

$$\Phi_v(\omega) = \lambda \sum_{k=0}^{\infty} h(k)e^{-ik\omega} \sum_{s=0}^{\infty} h(s)e^{i\omega s} \tag{111}$$

$$= \lambda H(e^{i\omega})H(e^{-i\omega}) \tag{112}$$

$$= \lambda |H(e^{i\omega})|^2 \tag{113}$$

It is easy to show that if a signal $\{s(t)\}$ is composed of a deterministic part $\{u(t)\}$ with spectrum $\Phi_u(\omega)$ and a stationary stochastic part $\{v(t)\}$ with zero mean and spectrum $\Phi_v(\omega)$ then the spectrum of $\{s(t)\}$ is given by:

$$\Phi_s(\omega) = \Phi_u(\omega) + \Phi_v(\omega) \tag{114}$$

1.11. Transformation of Spectra in Linear Systems

We will present equations for the general case of a multiple input, multiple output system (MIMO). Suppose a system has m inputs all collected in a vector $u(t)$ and p outputs all collected in a vector $y(t)$. The disturbances are also described by a p dimensional vector $e(t)$. Let the system be described by:

$$y(t) = G(q)u(t) + H(q)e(t) \tag{115}$$

where $G(q)$ is a p by m matrix and $H(q)$ is a p by p matrix. The covariance of the vector $u(t)$ would be defined as

$$R_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Eu(t)u^T(t-\tau) \tag{116}$$

The covariance of the vector $e(t)$ would be given as

$$R_e(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Ee(t)e^T(t-\tau) \tag{117}$$

Since $\{e(t)\}$ is a vector of independent and random variables

$$R_e(\tau) = \begin{cases} 0 & \text{if } \tau \neq 0 \\ \Lambda & \text{if } \tau = 0 \end{cases} \tag{118}$$

where Λ is a diagonal matrix of the variances of the individual components $e_i(t)$ of $e(t)$. Note that now $G(q)$ and $H(q)$ can be written as

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k} \tag{119}$$

$$H(q) = \sum_{k=0}^{\infty} h(k) q^{-k} \tag{120}$$

where $g(k)$ is a p by m matrix and $h(k)$ is a p by p matrix. Now suppose a process $\{y_1(t)\}$ is given as

$$y_1(t) = G(q)u(t) \tag{121}$$

The spectrum of $\{y_1(t)\}$ would be given by

$$\begin{aligned}
\Phi_{y_1}(\omega) &= \sum_{\tau=-\infty}^{\infty} R_{y_1}(\tau) e^{-i\tau\omega} && (122)\\
&= \sum_{\tau=-\infty}^{\infty} \left\{ \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left[\sum_{k=1}^{\infty} g(k) u(t-k) \right] \right. \\
&\qquad \left. \left[\sum_{m=1}^{\infty} u^T(t-\tau-m) g^T(m) \right] \right\} e^{-i\tau\omega} \\
&= \sum_{k=1}^{\infty} g(k) \sum_{m=1}^{\infty} \sum_{\tau=-\infty}^{\infty} R_u(\tau+m-k) e^{-i(\tau+m-k)\omega} e^{im\omega} e^{-ik\omega} && (123)\\
&= \sum_{k=1}^{\infty} g(k) e^{-ik\omega} \sum_{s=-\infty}^{\infty} R_u(s) e^{-is\omega} \sum_{m=1}^{\infty} g^T(m) e^{im\omega} && (124)\\
&= G(e^{i\omega}) \Phi_u(\omega) G^T(e^{-i\omega}) && (125)
\end{aligned}$$

Similarly if $\{y_2(t)\}$ is given by

$$y_2(t) = H(q)e(t) \tag{126}$$

then

$$\Phi_{y_2}(\omega) = H(e^{i\omega}) \Phi_e(\omega) H^T(e^{-i\omega}) \tag{127}$$

But

$$\Phi_e(\omega) = \sum_{\tau=-\infty}^{\infty} R_e(\tau) e^{-i\tau\omega} \tag{128}$$

From equations (118) and (128) we see that

$$\Phi_e(\omega) = \Lambda \tag{129}$$

Thus

$$\Phi_{y},\ (\omega) = H(e^{i\omega}) \Lambda H^T(e^{-i\omega}) \tag{130}$$

From the definition of $y_1(t)$ and y_2(t)

$$y(t) = y_1(t) + y_2(t)$$

Since $e(t)$ is a sequence of independent random variables with zero mean value

$$\Phi_y(\omega) = \Phi_{y_1}(\omega) + \Phi_{y_2}(\omega) \tag{131}$$

From equations (122) and (130)

$$\Phi_y(\omega) = G(e^{i\omega})\Phi_u(\omega)G^T(e^{-i\omega}) + H(e^{i\omega})\Lambda H^T(e^{-i\omega}) \tag{132}$$

This is a very important result which will be used later.

1.12. Invertibility of the Noise Model

According to our "lumped" formulation we consider the disturbance to be a filtered white noise

$$v(t) = H(q)e(t) \tag{133}$$

We will require the above noise model to be invertible. This means that given $v(k)$, $k \leq t$ we should be able to compute $e(k)$, $k \leq t$. Suppose $H(z) = \sum_{k=0}^{\infty} h(k)z^{-k}$ is stable and assume that $\frac{1}{H(z)}$ is analytic in $|z| \geq 1$.

$$\frac{1}{H(z)} = \sum_{k=0}^{\infty} \tilde{h}(k)z^{-k} \tag{134}$$

We define the filter $H^{-1}(q)$ by

$$H^{1}_{(q)} = \sum_{k=0}^{\infty} \tilde{h}(k)q^{-k} \tag{135}$$

Assume that the above filter is stable, that is:

$$\sum_{k=0}^{\infty} |\tilde{h}(k)| < \infty$$

Now

$$H(q)\frac{1}{H(q)} = \sum_{k=0}^{\infty} h(k)q^{-k} \sum_{s=0}^{\infty} \tilde{h}(s)q^{-s} \tag{136}$$

$$1 = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} h(k)\tilde{h}(s)q^{-(k+s)} \tag{137}$$

$$= \sum_{s=0}^{\infty}\sum_{m=s}^{\infty} h(m-s)\tilde{h}(s)q^{-m} \tag{138}$$

$$= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} h(m-s)\tilde{h}(s)q^{-m} \tag{139}$$

Equating powers of q on both sides of the above equation:

$$\sum_{s=0}^{m} \tilde{h}(s)h(m-s) = \begin{cases} 1 & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m \neq 0 \end{cases} \tag{140}$$

Now
$$\frac{1}{H(q)}v(t) = \sum_{s=0}^{\infty} \tilde{h}(s)v(t-s) \tag{141}$$
$$= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} h(k)e(t-s-k) \tag{142}$$
$$= \sum_{s=0}^{\infty} \tilde{h}(s) \sum_{m=s}^{\infty} h(m-s)e(t-m) \tag{143}$$
$$= \sum_{s=0}^{\infty}\sum_{m=s}^{\infty} \tilde{h}(s)h(m-s)e(t-m) \tag{144}$$
$$= \sum_{m=0}^{\infty} \left[\sum_{s=0}^{\infty} \tilde{h}(s)h(m-s)\right] e(t-m) \tag{145}$$

From equation (140), we get

$$\frac{1}{H(q)}v(t) = e(t) \tag{146}$$

1.13. One Step Ahead Predictor for $v(t)$

We now consider the question of estimating $v(t)$ given information on $v(s)$ for $s \leq t - 1$. We know that $v(t)$ is given by

$$v(t) = \sum_{k=0}^{\infty} h(k)e(t-k)$$
$$= h(0)e(t) + \sum_{k=1}^{\infty} h(k)e(t-k)$$

Assuming that $H(q)$ is monic, that is $h(0) = 1$, we get

$$v(t) = e(t) + \sum_{k=1}^{\infty} h(k)e(t-k) \tag{147}$$

With the assumption of invertibility of the noise model, we know the values of $e(s)$ for s $\leq t - 1$. This means that the second term in equation (147) is known at time $t - 1$. Thus the predicted value $\hat{v}(t|t-1)$ of $v(t)$ given $v(s)$, $s \leq t - 1$ is

$$\hat{v}(t|t-1) = Ee(t) + \sum_{k=1}^{\infty} h(k)e(t-k) \tag{148}$$

Since $e(t)$ is a random sequence with zero mean

$$\hat{v}(t|t-1) = \sum_{k=1}^{\infty} h(k)e(t-k) \tag{149}$$

Writing the above equation in terms of $H(q)$, we see that

$$\hat{v}(t|t-1) = [H(q) - 1]e(t) \tag{150}$$

1.14. One Step Ahead Predictor for $y(t)$

Suppose that the system output $y(s)$, $s \le t-1$ and $u(s)$, $\le t-1$ are known. If $y(t)$ is related to $u(t)$ and the noise $v(t)$ by the model

$$y(t) = G(q)u(t) + v(t) \tag{151}$$

then $v(s)$, $s \le t-1$ is also known at time t. We wish to estimate the value of $y(t)$ under these conditions. The one step ahead predictor for $v(t)$ gives

$$\hat{v}(t|t-1) = [H(q) - 1]\, e(t) \tag{152}$$

Assuming that the noise model $H(q)$ is invertible

$$e(t) = \frac{1}{H(q)} v(t) \tag{153}$$

Substituting equation (153) in equation (152), we get

$$\hat{v}(t|t-1) = \left[1 - H^{-1}(q)\right] v(t) \tag{154}$$

But from equation (151) $v(t)$ is given by

$$v(t) = y(t) - G(q)u(t) \tag{155}$$

Substituting equation (155) in (154), we get

$$\hat{v}(t|t-1) = \left[1 - H^{-1}(q)\right] [y(t) - G(q)u(t)] \tag{156}$$

The one step ahead prediction of $y(t)$ is then given by

$$\hat{y}(t|t-1) = G(q)u(t) + \hat{v}(t|t-1) \tag{157}$$

Substituting (156) in (157), we get

$$\hat{y}(t|t-1) = H^{-1}(g)G(q)u(t) + \left[1 - H^{-1}(q)\right] y(t) \tag{158}$$

In matrix form

$$\hat{y}(t|t-1) = W(q)z(t) \tag{159}$$

where $$W(q) = \left[H^{-1}(q)G(q)1 - H^{-1}(q)\right]$$

and $$z(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$$

1.15. Equality of two Models

Two stable predictor models $W_1(q)$ and $W_2(q)$ are said to be “equal” if

$$W_1(e^{i\omega}) = W_2(e^{i\omega}), \text{ for almost all } \omega \tag{160}$$

Choice of $G(q)$ and $H(q)$ leads to a variety of model structures (see [7], [14]). For input design one often uses "noise free" predictors in which case $v(t) \equiv 0$ and consequently $z(t) \equiv u(t)$ and $W(q) \equiv G(q)$.

1.16. Prediction Error Approach

Selection of $G(q)$ and $H(q)$ as rational functions results in the specification of a model structure $\mathcal{M}$. There are infinitely many models in $\mathcal{M}$ depending on how $G(q)$ and $H(q)$ are parameterized. Let θ denote the vector of all the parameters in $G(q)$ and $H(q)$. The dependance of $G(q)$ and $H(q)$ on θ will be indicated by writing out $G(q)$ and $H(q)$ as $G(q, \theta)$ and $H(q, \theta)$ respectively. The model from the model structure $\mathcal{M}$ corresponding to the parameter vector θ will be denoted by $\mathcal{M}(\theta)$.

Suppose $\theta \in \mathcal{D}_M \subset R^d$ where d is the dimension of θ and let

$$\mathcal{M}^* = \{\mathcal{M}(\theta) | \theta \in \mathcal{D}_M\}$$

To decide which model $\mathcal{M}(\theta)$ in $\mathcal{M}^*$ best represents the collected data

$$Z^N = \{y(1), y(2), \dots, y(N), u(1), u(2), \dots, u(N)\}$$

we need a method of evaluating the model $\mathcal{M}(\theta)$. Let $\hat{y}(t|\theta)$ be the predictor model corresponding to $\mathcal{M}(\theta)$. Define the predictor error

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t|\theta) \tag{161}$$

Given, Z^N, one can compute $\varepsilon(t, \theta)$ for $t = 1, 2, \dots, N$. A common approach to selecting the best model from the set $\mathcal{M}^*$ is to select $\theta_N \in \mathcal{D}_M$ such that

$$\theta_N = \arg \min_{\theta \in \mathcal{D}_M} V_N(\theta, Z^N) \tag{162}$$

where
$$V_N(\theta, Z^N) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} \varepsilon^2(t, \theta) \tag{163}$$

1.17. Informative Experiments

One obvious requirement on the data used for system identification is that the data should be able to distinguish between two models $\mathcal{M}(\theta_1)$ and $\mathcal{M}(\theta_2)$ from the model set $\mathcal{M}$. Consider two models corresponding to $\mathcal{M}(\theta_1)$ and $\mathcal{M}(\theta_2)$

$$y(t|\theta_1) = G_1(q)u(t) + H_1(q)e(t) \tag{164}$$

and
$$y(t|\theta_2) = G_2(q)u(t) + H_2(q)e(t) \tag{165}$$

Denote the difference in predictions of these models by $\Delta\varepsilon(t)$

$$\begin{aligned}\Delta\varepsilon(t) &= y(t|\theta_1) - y(t|\theta_2)\\ &= [G_1(q) - G_2(q)]u(t) + [H_1(q) - H_2(q)]e(t)\\ &= \Delta G(q)u(t) + \Delta H(q)e(t)\end{aligned}$$

Using equation (105) and assuming the $u(t)$ and $e(t)$ are uncorrelated and that $e(t)$ has variance λ_0

$$\overline{E}\Delta\varepsilon^2(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[|\Delta G(e^{i\omega})|^2\Phi_u(\omega) + |\Delta H(e^{i\omega})|^2\lambda_0\right]d\omega \tag{166}$$

Now suppose the data are not sufficiently informative enough. Then they cannot distinguish between the models $\mathcal{M}(\theta_1)$ and $\mathcal{M}(\theta_2)$. Thus

$$\overline{E}\Delta\varepsilon^2(t) = 0 \tag{167}$$

From equations (166) and (167), we get

$$\Delta H(e^{i\omega}) = 0$$

and

$$|\Delta G(e^{i\omega})|^2\Phi_u(\omega) = 0 \tag{168}$$

If the above equation implies that $\Delta G(e^{i\omega}) = 0$ for almost all ω then by definition of model equality

$$G_1(q) \equiv G_2(q) \tag{169}$$

which means that the data are sufficiently informative w.r.t. $\mathcal{M}$. Basically if $\Phi_u(\omega) > 0$ at "sufficiently many" ω then

$$|\Delta G(e^{i\omega})|^2\Phi_u(\omega) = 0 \Rightarrow \Delta G(e^{i\omega}) = 0 \text{ for almost all } \omega \tag{170}$$

At how many points do we need $\Phi_u(\omega) > 0$? The answer lies in a related concept called persistence of excitation (see [14], [7]). A quasi stationary signal $u(t)$ is said to be persistently exciting of order n if for all filters of order n

$$M_n(q) = m_1q^{-1} + \cdots + m_nq^{-n}$$

$$|M_n(e^{i\omega})|^2\Phi_u(\omega) \Rightarrow M_n(e^{i\omega}) = 0 \tag{171}$$

Since there are n unknowns in $M_n(q)$, if $\Phi_u(\omega) > 0$ at n points then

$$M_n(e^{i\omega^k}) = 0, \quad k = 1, 2, \ldots, n$$

Thus, m_1, m_2, $\ldots$,m$_n$= 0 and so $M_n(e^{i\omega}) = 0$. Therefore, we need

$$\Phi_u(\omega) > 0$$

at n points. Consider a set $\mathcal{M}$ of models where

$$G(q,\theta) = \frac{B(q,\theta)}{F(q,\theta)} \tag{172}$$

where $B(q, \theta)$ has degree n_b and $F(q, \theta)$ has degree n_f. For any two models $G_1(q)$ and $G_2(q)$ in this set

$$\Delta G(q) = \frac{B_1(q)F_2(q) - B_2(q)F_1(q)}{F_1(q)F_2(q)} \tag{173}$$

Thus from

$$|\Delta G(e^{i\omega})|^2 \Phi_u(\omega) = 0 \tag{174}$$

we get

$$\frac{|B_1(e^{i\omega})F_2(e^{i\omega}) - B_2(e^{i\omega})F_1(e^{i\omega})|^2}{|F_1(e^{i\omega})|^2|F_2(e^{i\omega})|^2}\Phi_u(\omega) = 0 \tag{175}$$

or

$$|B_1(e^{i\omega})F_2(e^{i\omega}) - B_2(e^{i\omega})F_1(e^{i\omega})|^2 \Phi_u\omega = 0 \tag{176}$$

Now the polynomial $\left[B_1(e^{i\omega})F_2(e^{i\omega}) - B_2(e^{i\omega})F_1(e^{i\omega})\right]$ has $n_b + n_f$ unknowns. Thus the input should be persistently exciting of order $n_b + n_f$. Thus

$$\Phi_u(\omega) > 0 \text{ at } n_b + n_f \text{ points} \tag{177}$$

Thus the degree of the polynomial in the numerator of $\Delta G(q)$ is important.

1.18. Persistence of Excitation for Multivariable Systems

We provide the following original treatment to the issue of persistence of excitation for multivariable systems. Consider a multiple-input multiple-output (MIMO) system with m inputs expressed in an input vector $u(t)$ and a p dimensional output expressed in an output vector $y(t)$. Suppose the system is modelled by two different models $\mathcal{M}(\theta_1)$ and $\mathcal{M}(\theta_2)$ as follows:

$$y(t|\theta_1) = G_1(q)u(t) + H_1(q)e(t) \tag{178}$$

and

$$y(t|\theta_2) = G_2(q)u(t) + H_2(q)e(t) \tag{179}$$

where $e \in R^p$, $H_1,\ H_2 \in R^{p\times p}$ and $G_1,\ G_2 \in R^{p\times p}$. The error in prediction between these two models is

$$\varepsilon(t) = \Delta G(q)u(t) + \Delta H(q)e(t) \tag{180}$$

where

$$\Delta G(q) = G_1(q) - G_2(q) \tag{181}$$

and

$$\Delta H(q) = H_1(q) - H_2(q) \tag{182}$$

The jth component of $\varepsilon(t)$ is

$$\varepsilon_j(t) = [\Delta G(q)]_j\, u(t) + [\Delta H]_j\, e(t) \tag{183}$$

Here $[\Delta G(q)]_j$ is the jth row of the matrix $\Delta G(q)$ and $[\Delta H(q)]_j$ is the jth row of the matrix $\Delta H(q)$. Let

$$[\Delta G(q)]_j = [g_{j1}(q), g_{j2}(q), \ldots, g_{jm}(q)] = g_j(q) \tag{184}$$

and $$[\Delta H(q)]_j = [h_{j1}(q), h_{j2}(q), \dots, h_{j_m}(q)] = h_j(q) \tag{185}$$

From (105) and (132) and the usual assumptions about $e(t)$, we get

$$\overline{E}_e^2(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_j(e^{i\omega})\Phi_u(\omega)g_j^T(e^{-i\omega})d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} h_j(e^{\iota\omega})\Lambda_j^T(e^{-\iota\omega})d\omega \tag{186}$$

Suppose the two models described above give the same predictions and

$$\overline{E}\varepsilon_j^2(t) = 0 \tag{187}$$

Note that the signal

$$s(t) = h_j(q)e(t) \tag{188}$$

has the spectrum

$$\Phi_s(\omega) = h_j(e^{i\omega})\Lambda h_j^T(e^{-i\omega}) \tag{189}$$

and the signal

$$\omega(t) = g_j(q)u(t) \tag{190}$$

has the spectrum

$$\Phi_w(\omega) = g_j(e^{i\omega})\Phi_u(\omega)g_j^T(e^{-i\omega}) \tag{191}$$

Since the spectrum of any signal is always non-negative

$$\Phi_s(\omega) = h_j(e^{i\omega})\Lambda h_j^T(e^{-i\omega}) \geq 0 \tag{192}$$

and $$\Phi_w(\omega) = g_j(e^{i\omega})\Phi_u(\omega)g_j^T(e^{-i\omega}) \geq 0 \tag{193}$$

From (192), (193), (186) and (187)

$$h_j(e^{i\omega})\Lambda h_j^T(e^{-i\omega}) = 0 \tag{194}$$

and $$g_j(e^{i\omega})\Phi_u(\omega)g_j^T(e^{-i\omega}) = 0 \tag{195}$$

If the above equation implies $g_j(q) = 0$, then the two models will be equal. From (194), since Λ is diagonal

$$h_j(e^{i\omega}) = 0 \ \forall \ \omega \tag{196}$$

This means that

$$H_1(q) = H_2(q) \tag{197}$$

Suppose the maximum numerator filter order of all the terms in $\Delta G(q)$ is n and suppose that $\Phi_u(\omega)$ is positive definite at n distinct frequencies $\omega_1, \omega_2, \dots, \omega_n$.

$$x^*\Phi_u(\omega_r)x > \ 0 \ \forall x \in R^m - \{0\} \text{ and } r = 1, 2, \dots, n \tag{198}$$

From (195) and (198) we get

$$g_j(e^{i\omega^r}) = 0 \quad r = 1, 2, \dots, n \tag{199}$$

Let
$$g_j(q) = [g_{j1}(q), g_{j2}(q), \ldots, g_{jk}(q), \ldots, g_{jm}(q)] \tag{200}$$

Take a representative member from the above vector, say $g_{jk}(q)$. The maximum numerator order of any filter in $\Delta G(q)$ is n and so we can write

$$\hat{g}_{jk}(q) = m_1 q^{-1} + \ldots + m_n q^{-n} \tag{201}$$

where $\hat{g}_{jk}(q)$ is the numerator of $g_{jk}(q)$. From (199), (200) and (201)

$$\hat{g}_{jk}(e^{i\omega^r}) = m_1 e^{-i\omega^r} + \ldots + m_n e^{in\omega^r} = 0, \quad r = 1, 2, \ldots, n \tag{202}$$

The above system of equations can be written more compactly as

$$V \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = 0 \tag{203}$$

where V is the matrix

$$V = \begin{bmatrix} e^{-i\omega_1} & \ldots & e^{-in\omega_1} \\ \vdots & \ddots & \vdots \\ e^{-i\omega_n} & \ldots & e^{in\omega_n} \end{bmatrix} \tag{204}$$

On close inspection, we see that V is actually a vandermonde matrix and is therefore nonsingular. Therefore, (203) implies

$$\begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = 0 \tag{205}$$

From (205) and (201)

$$\hat{g}_{jk}(q) = 0 \text{ and so } g_{jk}(q) = 0 \tag{206}$$

Extending the above reasoning to $k = 1, \ldots, m$ and $j = 1, \ldots, p$ we see that

$$g_j(q) = 0, \quad j = 1, \ldots, p \tag{207}$$

This means that

$$\Delta G(q) = G_1(q) - G_2(q) = 0 \tag{208}$$

Combined with (197) we deduce that and the two models are equal. Let

$$n = \text{ max numerator filter order in } \Delta G(q)$$

We thus see that if $\Phi_u(\omega)$ is positive definite at n distinct frequencies then the input is able to distinguish between two distinct models and is therefore sufficiently informative.

An m dimensional input $u(t)$ is said to be persistently exciting of order n if the $m \times n$ matrix $\Phi_u(\omega)$ is positive definite at n distinct frequencies. The maximum numerator order of a filter permitted in the matrix $G(q)$ for an informative experiment in this case is n. Since $\Phi_u(\omega)$ is already symmetric, we require the minimum singular value of $\Phi_u(\omega)$ to be positive at n frequencies. Mathematically,

$$\sigma_{\min}(\Phi_u(\omega_r)) > 0 \text{ for } r = 1, 2, \ldots, n$$

2. Weyl Framework for Input Signal Design

2.1. Input Signal Design

In system identification, we are interested in modeling the system by a particular model set around the current operating point. An identification test consists of the introduction of an input signal $u(t)$ into the system and measurement of the system output $y(t)$. The prediction error approach described before is then used to obtain the unknown parameters in the proposed model.

It is interesting to note that one cannot introduce just any arbitrary input $u(t)$ into the system. One has to take into account various practical and theoretical requirements:

1. The input $u(t)$ should satisfy the theoretical requirement of persistence of excitation so that the data collected are able to distinguish between different models in the model set.
2. The input should be designed in a way so that $|u(t)| \leq C_1$ and/or $|u(t+1) - u(t)| \leq C_2$ where C_1 and C_2 are constants. This means that the inputs should satisfy bound and/or move-size constraints. This is reasonable in view of the fact that identification tests are usually conducted when plant is in operation and so one usually does not want large fluctuations in $u(t)$. Given the fact that the outputs normally have to meet certain specifications, one would want to impose the bound and/or move size constraints on the outputs also.

When these requirements are met the input $u(t)$ is said to be "plant friendly" (see [14])

Another consideration for multiple input multiple output (MIMO) systems is that the output state space should have good directionality information for implementing advanced control philosophies like Model on Demand (MoD). This means that the output should not fall in only one favored direction(s). This requirement is particularly difficult to meet for highly interactive systems as we will see later.

For simple linear systems with known steady state gain matrices, one can use the singular value information from an SVD to design inputs via scaling to excite the low gain direction (see [5]).

Unlike this simplistic approach, we will develop a powerful framework for input signal design that can excite all directions including the low gain direction and that has the following features:

1. Is applicable to both linear as well as non-linear process systems.
2. This approach is not limited to bivariate MIMO systems. Theoretically speaking, this approach is applicable to higher dimensional MIMO systems also.
3. As an added advantage, this approach allows for the enforcement of a variety of "plant friendliness" constraints.
4. Is based on a solid mathematical principle called the Weyl criterion.

2.2. Multisine Input Signals

The input signals used for plant friendly system identification in this study will be multisine signals which are essentially a linear combination of sinusoids of different frequencies [13].

$$u(t) = \sum_{j=1}^{n_s} \sqrt{2\alpha_j} \cos(\omega_j t + \phi_j) \tag{209}$$

Choosing a sampling time of T and setting $t = kT$

$$u(k) = \sum_{j=1}^{n_s} \sqrt{2\alpha_j} \cos(\omega_j kT + \phi_j) \tag{210}$$

We want the signal $u(k)$ to be periodic with period N_s. This is achieved by setting

$$\omega_j = \frac{2\pi j}{N_s T} \tag{211}$$

In view of the Shannon's Sampling Theorem, the maximum allowable frequency in the input is $\frac{\pi}{T}$, the Nyquist frequency. This means that

$$n_s \leq \frac{N_s}{2} \tag{212}$$

It will be necessary to calculate the spectrum $\Phi_u(\omega)$ of the above signal $u(k)$.

2.3. Spectrum of a Sinusoid

First consider any sinusoid from $u(t)$, say

$$w(k) = \sqrt{2\alpha_j} \cos(\omega_j kT + \phi_j) = \sqrt{2\alpha_j} \cos\left(\frac{2\pi j}{N_s} k + \phi_j\right) \tag{213}$$

The auto-covariance of $w(k)$ is

$$R_w(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} w(k) w(k-\tau) \tag{214}$$

$$= \alpha_j \left\{ \cos\left(\frac{2\pi j \tau}{N_s}\right) + \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \cos(\hat{\omega} k + \phi) \right\} \tag{215}$$

$$\tag{216}$$

In equation (215)

$$\hat{\omega} = \frac{4\pi j}{N_s} \tag{217}$$

and

$$\phi = 2\phi_j - \frac{2\pi j\tau}{N_s} \tag{218}$$

Note that $\cos(\hat{\omega}k + \phi)$ has period $M = \frac{N_s}{2}$.

Let K be the integer such that $KM \leq N$ and $KM + M > N$. Now

$$\sum_{k=1}^{N} \cos(\hat{\omega}k + \phi) = MK \left\{ \frac{1}{M} \sum_{k=1}^{M} \cos(\hat{\omega}k + \phi) \right\} + \sum_{k=KM}^{N} \cos(\hat{\omega}k + \phi) \tag{219}$$

The second term has at most $(M-1)$ terms and is therefore bounded. Hence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \cos(\hat{\omega}k + \phi) = \frac{1}{M} \sum_{k=1}^{M} \cos(\hat{\omega}k + \phi) \tag{220}$$

Since $M = \dfrac{N_s}{2}$

$$\frac{1}{M} \sum_{k=1}^{M} \cos(\hat{\omega}k + \phi) = \frac{2}{N_s} \sum_{k=1}^{\frac{N_s}{2}} \cos(\hat{\omega}k + \phi) \tag{221}$$

$$= \frac{2}{N_s} \mathrm{Re} \left\{ e^{i\phi} \sum_{k=1}^{\frac{N_s}{2}} e^{i\hat{\omega}k} \right\} \tag{222}$$

$$= 0 \tag{223}$$

In (222) we have used the identity

$$\sum_{k=1}^{\frac{N_s}{2}} e^{i\hat{\omega}k} = 0, \text{ if } \hat{\omega} = \frac{4\pi j}{N_s} \tag{224}$$

From (215), (220) and (223)

$$R_w(\tau) = \alpha_j \cos\left(\frac{2\pi j\tau}{N_s}\right) \tag{225}$$

The power spectrum $\Phi_w(\omega)$ is

$$\Phi_w(\omega) = \sum_{\tau=-\infty}^{\infty} R_w(\tau) e^{-i\tau\omega T} \tag{226}$$

$$= \frac{\alpha_j}{2} \sum_{\tau=-\infty}^{\infty} \left\{ e^{-i\tau T(\omega-\omega_j)} + e^{-i\tau T(\omega+\omega_j)} \right\} \tag{227}$$

Note that

$$\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \sum_{\tau=-\infty}^{\infty} \left[e^{-i\tau T(\omega-\omega_j)} + e^{-i\tau T(\omega+\omega_j)}\right] d\omega = \frac{4\pi}{T} \tag{228}$$

while

$$\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{2\pi}{T} \left[\delta(\omega-\omega_j) + \delta(\omega+\omega_j)\right] d\omega = \frac{4\pi}{T} \tag{229}$$

Here $\delta(.)$ is the Dirac Delta function.

In view of (228) and (229) we can write (227) as

$$\Phi_w(\omega) = \frac{\alpha_j}{2}\frac{2\pi}{T}\left[\delta(\omega-\omega_j) + \delta(\omega+\omega_j)\right] \tag{230}$$

$$= \frac{\pi\alpha_j}{T}\left[\delta(\omega-\omega_j) + \delta(\omega+\omega_j)\right] \tag{231}$$

Note that the Parseval's relationship holds:

$$\frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \Phi_w(\omega)d\omega = R_w(0) = \alpha_j$$

Thus the spectrum of $\sqrt{2\alpha_j}\cos(\omega_j kT + \phi_j) = \sqrt{2\alpha_j}\cos\left(\frac{2\pi j}{N_s}k + \phi_j\right)$ has two peaks: at $+\dfrac{2\pi j}{N_s T}$ and at $\dfrac{-2\pi j}{N_s T}$.

2.4. Cross Spectrum between two Sinusoids

Consider two sinusoids

$$w_1(k) = \sqrt{2\alpha_p}\cos\left(\frac{2\pi p}{N_s}k + \phi_p\right) \tag{232}$$

and

$$w_2(k) = \sqrt{2\alpha_q}\cos\left(\frac{2\pi q}{N_s}k + \phi_q\right) \tag{233}$$

Consider first the case where $w_1$and w_2 are uncorrelated, i.e., $p \neq q$. The cross covariance is

$$R_{w_1w_2}(\tau) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} 2\sqrt{\alpha_p\alpha_q}\cos\left(\frac{2\pi p}{N_s}k + \phi_p\right)\cos\left(\frac{2\pi q}{N_s}k + \phi_q - \frac{2\pi q}{N_s}\tau\right) \tag{234}$$

On simplification,

$$R_{w_1w_2}(\tau) = \sqrt{\alpha_p\alpha_q}\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\left[\cos\left(\frac{2\pi(p+q)}{N_s}k + \phi_A\right) + \cos\left(\frac{2\pi(p-q)}{N_s}k + \phi_B\right)\right] \tag{235}$$

where

$$\phi_A = \phi_p + \phi_q - \frac{2\pi q\tau}{N_s} \tag{236}$$

and
$$\phi_B = \phi_p - \phi_q + \frac{2\pi q\tau}{N_s} \tag{237}$$

Since $p \neq q$ the discussion from (215) to (224) implies

$$R_{w_1w_2}(\tau) = 0 \text{ if } p \neq q \tag{238}$$

This means that

$$\Phi_{w_1w_2}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{w_1w_2}(\tau)e^{-i\tau\omega T} = 0 \;\forall\; \omega \text{ if } p \neq q \tag{239}$$

Next, consider the case when w_1 and w_2 are correlated, i.e., $\frac{2\pi p}{N_sT} = \frac{2\pi q}{N_sT}$. In this case the first term on the right hand side in (235) still goes to zero from (215) to (224). Then

$$R_{w_1w_2}(\tau) = \sqrt{\alpha_p\alpha_q}\cos\left(\frac{2\pi q}{N_s}\tau + \phi_p - \phi_q\right) \tag{240}$$

By an argument similar to that in (226) to (229) we can show that when $p = q$

$$\Phi_{w_1w_2}(\omega) = \frac{\pi\sqrt{\alpha_p\alpha_q}}{T}\cos(\phi_p - \phi_q)\left[\delta(\omega - \frac{2\pi q}{N_sT}) + \delta(\omega + \frac{2\pi q}{N_sT})\right] \tag{241}$$

It is interesting to note that the cross-spectrum can be made zero or non-zero not only by the selection of amplitudes α_p and α_q but also by the selection of the phase difference $(\phi_p - \phi_q)$ between the two sinusoids.

2.5. Multichannel Multisine Input

The case studies undertaken in this work will be 2-input 2-output systems. Hence the input $u(k)$ used for identification will be a vector

$$u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \tag{242}$$

where $u_1(k)$ and $u_2(k)$ are defined as

$$u_1(k) = \sum_{p=1}^{n_s} \sqrt{2\alpha_{p1}}\cos(\frac{2\pi p}{N_s}k + \phi_{p1}) \tag{243}$$

$$u_2(k) = \sum_{q=1}^{n_s} \sqrt{2\alpha_{q2}}\cos(\frac{2\pi q}{N_s}k + \phi_{q2}) \tag{244}$$

The design variables include the amplitudes and phases of the two signals as well as specification of the signal length N_s and the number of sinusoids n_s. Increasing n_s leads to inclusion of sinusoids of high frequency and increasing N_s leads to the inclusion of sinusoids of low frequency. These parameters are decided on the basis of the frequency range of interest [6].

The amplitudes and phases are selected using the Weyl optimization framework which will be described shortly.

2.6. Persistence of Excitation

We noted before that the theoretical requirement of persistence of excitation should be satisfied for an informative data set. For multisine signals it is very easy to achieve this objective using a so called "zippered" power spectrum (Rivera et al. 1997). This means that the coefficients α_{p_1} and α_{q_2} in (243) and (244) are chosen to be alternately zero and non-zero respectively.

$$\alpha_{11} \neq 0, \quad \alpha_{21} = 0, \quad \alpha_{31} \neq 0, \quad \alpha_{41} = 0 \dots \tag{245}$$

and
$$\alpha_{12} = 0, \quad \alpha_{22} \neq 0, \quad \alpha_{32} = 0, \quad \alpha_{42} \neq 0 \dots \tag{246}$$

Using (231) and (240) and given the fact that we are using a "zippered" power spectrum:

$$\Phi_{u_1}(\omega) = \sum_{p=1}^{n_s} \frac{\pi \alpha_{p1}}{T} \left[\delta(\omega - \frac{2\pi p}{N_s T}) + \delta(\omega + \frac{2\pi p}{N_s T}) \right] \tag{247}$$

and
$$\Phi_{u_2}(\omega) = \sum_{q=1}^{n_s} \frac{\pi \alpha_{q2}}{T} \left[\delta(\omega - \frac{2\pi q}{N_s T}) + \delta(\omega + \frac{2\pi q}{N_s T}) \right] \tag{248}$$

Also
$$\Phi_{u_1 u_2}(\omega) = 0 \ \forall \ \omega \tag{249}$$

$$\Phi_{u_2 u_1}(\omega) = 0 \ \forall \ \omega \tag{250}$$

The input power spectrum is

$$\Phi_u(\omega) = \begin{pmatrix} \Phi_{u_1} & (\omega)\Phi_{u_1 u_2}(\omega) \\ \Phi_{u_2 u_1} & (\omega)\Phi_{u_2}(\omega) \end{pmatrix} \tag{251}$$

In view of (247)-(250)

$$\Phi_u(\omega) = \begin{pmatrix} \Phi_{u_1}(\omega) & 0 \\ 0 & \Phi_{u_2}(\omega) \end{pmatrix} \tag{252}$$

Suppose the maximum numerator filter order in $\Delta G(q)$ is n [see (208), (178) and (179)]. Consider

$$x(q) = [x_1(q), x_2(q)] \tag{253}$$

which is a row vector of two filters whose orders are at most n. In light of the discussion in Chapter 1, we must show that

$$x(e^{i\omega})\Phi_u(\omega)x^*(e^{i\omega}) = 0 \tag{254}$$

implies
$$x = 0 \tag{255}$$

Since the input signal has a "zippered" spectrum, if $\Phi_{u_1}(\omega) \neq 0$ then $\Phi_{u_2}(\omega) = 0$ and vice versa. Thus the matrix $\Phi_u(\omega)$ has the form

$$\Phi_u(\omega) = \begin{bmatrix} \Phi_{u_1}(\omega) & 0 \\ 0 & 0 \end{bmatrix} \quad \omega = \frac{2\pi l}{N_s T}, l = 1, 3, 5, \ldots \tag{256}$$

or
$$\Phi_u(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_{u_2}(\omega) \end{bmatrix} \quad \omega = \frac{2\pi r}{N_s T}, r = 2, 4, 6, \ldots \tag{257}$$

Therefore

$$x(e^{i\omega})\Phi_u(\omega)x^*(e^{i\omega}) = \begin{cases} \Phi_{u_1}(\omega)|x_1(e^{i\omega})|^2 \text{ if } \omega = \dfrac{2\pi l}{N_s T}, l = 1, 3, 5, \ldots \\ \Phi_{u_2}(\omega)|x_2(e^{i\omega})|^2 \text{ if } \omega = \dfrac{2\pi r}{N_s T}, r = 2, 4, 6, \ldots \end{cases} \tag{258}$$

If Φ_{u_1} is non-zero at n frequencies then as shown in Chapter 1, the non-singularity of vandermonde matrix guarantees that

$$x_1(q) = 0 \tag{259}$$

Similarly, if Φ_{u_2} is non-zero at n frequencies then

$$x_2(q) = 0 \tag{260}$$

Since the frequencies at which Φ_{u_1} is non-zero are different from the frequencies at which Φ_{u_2} is zero, we need at least $2n$ frequencies in the input signal. Thus

$$n_s \geq 2n \tag{261}$$

If (261) is satisfied then the input signal would be persistently exciting for the "zippered spectrum" experiment.

2.7. Highly Interactive Systems: Special Considerations

The following is an original treatment of the issue of highly interactive systems and the challenges that they present. Suppose the output of a 2×2 is given by

$$y(k) = G(q)u(k) \tag{262}$$

where $y(k) \equiv y(kT)$ and $u(k) \equiv u(kT)$, T is the sampling time. $G(q)$ is a 2×2 matrix of transfer functions. We drop T for ease of notation. Since the systems we will be considering are linear, a linear combination of inputs produces a linear combination of outputs. Also since

$$\cos(\omega kT + \phi) = \frac{1}{2}\left[e^{i(\omega kT + \phi)} + e^{-i(\omega kT + \phi)}\right]$$

it is sufficient, for analysis purposes, to consider a general sinusoidal input at frequency ω of the form

$$u(k)\begin{bmatrix}u_1(k)\\u_2(k)\end{bmatrix} = \begin{bmatrix}\alpha_1 e^{i(\omega kT+\phi_1)}\\ \alpha_2 e^{i(\omega kT+\phi_2)}\end{bmatrix} \tag{263}$$

Here α_1 and α_2 are non-negative constants. Now

$$G(q) = \sum_{m=1}^{\infty} g(m)q^{-m} \tag{264}$$

where $g(m)$ is a 2×2 matrix and q is the shift operator. Therefore

$$y(k) = \sum_{m=1}^{\infty} g(m)q^{-m}u(k) \tag{265}$$

$$= \sum_{m=1}^{\infty} g(m)u(k-m) \tag{266}$$

$$= \sum_{m=1}^{\infty} g(m)\begin{bmatrix}\alpha_1 e^{i(\omega kT+\phi_1)}e^{-i\omega mT}\\ \alpha_2 e^{i(\omega kT+\phi_2)}e^{-i\omega mT}\end{bmatrix} \tag{267}$$

$$y(k) = G(e^{i\omega T})u(k) \tag{268}$$

Suppose the singular value decomposition (SVD) of $G(e^{i\omega T})$ is

$$G(e^{i\omega T}) = \left[w_a(e^{i\omega T})w_b(e^{i\omega T})\right]\begin{bmatrix}\sigma_1(\omega) & 0\\ 0 & \sigma_2(\omega)\end{bmatrix}\begin{bmatrix}v_a^*(e^{i\omega T})\\ v_b^*(e^{i\omega T})\end{bmatrix} \tag{269}$$

From now on, we will write $w_a\left(e^{i\omega T}\right)$, $w_b\left(e^{i\omega T}\right)$, $v_a\left(e^{i\omega T}\right)$, $v_a\left(e^{i\omega T}\right)$, $\sigma_1(\omega)$, $\sigma_2(\omega)$ as w_a, w_b, v_a, v_b, σ_1, σ_2 respectively for notational convenience. In (269), $\sigma_1(\omega)$ and $\sigma_2(\omega)$ are the real singular values and $\sigma_1 \geq \sigma_2 \geq 0$. Also w_a, w_b, v_a and v_b are column vectors and satisfy the orthogonality conditions:

$$\begin{bmatrix}w_a^*\\ w_b^*\end{bmatrix}\begin{bmatrix}w_a & w_b\end{bmatrix} = \begin{bmatrix}w_a^*w_a & w_a^*w_b\\ w_b^*w_a & w_b^*w_b\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{270}$$

and

$$\begin{bmatrix}v_a^*\\ v_b^*\end{bmatrix}\begin{bmatrix}v_a & v_b\end{bmatrix} = \begin{bmatrix}v_a^*v_a & v_a^*v_b\\ v_b^*v_a & v_b^*v_b\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{271}$$

The system described above is said to be "ill-conditioned" if σ_1 is considerably larger in magnitude as compared to σ_2.

$$\sigma_1 >> \sigma_2$$

To understand the effect of an ill-conditioned system consider the output $y(k)$ produced by a generalized sinusoidal input from equation (268):

$$y(k) = G(e^{i\omega T})u(k) \tag{272}$$

In view of (269), we can write

$$y(k) = \sigma_1 w_a v_a^* u(k) + \sigma_2 w_b v_b^* u(k) \tag{273}$$

If $\sigma_1 >> \sigma_2$, the first term will dominate in the above equation and we will have

$$y(k) \approx \sigma_1 w_a v_a^* u(k) \tag{274}$$

Noting that $v_a^* u(k)$ is a scalar of varying magnitude, we see that the output $y(k)$ is always in the same direction, namely w_a, the high gain direction. It is now clear that for highly interactive systems, special care must be given to input design in order to have good directionality information in the output. From (273), we see that to excite the low gain direction one would choose the input $u(k)$ in such a way that

$$K\sigma_1 v_a^* u(k) \approx \sigma_2 v_b^* u(k) \tag{275}$$

where K is a large positive constant. This would then compensate for the small value of σ_2 and produce an output predominantly in the low gain direction. Substituting $u(k)$ from (263) in (275) we get

$$K\sigma_1 v_a^* \begin{bmatrix} a_1 e^{i\phi_1} \\ a_2 e^{i\phi_2} \end{bmatrix} \approx \sigma_2 v_b^* \begin{bmatrix} a_1 e^{i\phi_1} \\ a_2 e^{i\phi_2} \end{bmatrix} \tag{276}$$

If we use an uncorrelated harmonic with say, $a_1 = 0$ then the above requirement reduces to

$$K\sigma_1 v_a^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \sigma_2 v_b^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{277}$$

which would hold only if we are very lucky! In order to ensure that (276) holds we need

$$\begin{bmatrix} a_1 e^{i\phi_1} \\ a_2 e^{i\phi_2} \end{bmatrix} = \alpha(\sigma_2 v_a + K\sigma_1 v_b) \tag{278}$$

where α is a scalar constant. Substituting (278) in (276) and noting the orthogonality conditions (271) we get

$$K\sigma_1 \alpha \sigma_2 \approx \sigma_2 \alpha K \sigma_1 \tag{279}$$

In order for (278) to hold, in general both a_1 and a_2 would be non-zero. This means that we would be using a correlated input. Since we are trying to estimate the model $G(q)$ in the first place, we cannot know beforehand the various directions w_a, w_b, v_a, v_b and the singular values σ_1 and σ_2. However the main point is that we should allow some input to be correlated for promoting output in the low gain direction as well. Also see [4], [10] and 5].

2.8. Modified Zippered Spectrum

In light of the above discussion, we will not use the standard zippered spectrum but also allow for correlated harmonics in the input. To do this we use a so called "modified" zippered spectrum. We modify the standard "zippered" spectrum (245),(246) as follows:

$$\alpha_{11} \neq 0, \alpha_{21} = 0, \alpha_{31} \neq 0, \alpha_{41} \neq 0, \alpha_{51} = 0, \alpha_{61} \neq 0, \dots \qquad (280)$$

and $$\alpha_{12} = 0, \alpha_{22} \neq 0, \alpha_{32} \neq 0, \alpha_{42} = 0, \alpha_{52} \neq 0, \alpha_{62} \neq 0, \dots \qquad (281)$$

It can be seen from (280) and (281) that every third harmonic is correlated. With this modified zippered spectrum we have the following power spectra:

$$\Phi_{u_1}(\omega_r) \neq 0, \Phi_{u_2}(\omega_r) = 0, \text{ and } \Phi_{u_1 u_2}(\omega_r) = 0, \text{ for } r = 1, 4, 7, \dots \qquad (282)$$

$$\Phi_{u_1}(\omega_r) = 0, \Phi_{u_2}(\omega_r) \neq 0, \text{ and } \Phi_{u_1 u_2}(\omega_r) = 0, \text{ for } r = 2, 5, 8, \dots \qquad (283)$$

and $$\Phi_{u_1}(\omega_r) \neq 0, \Phi_{u_2}(\omega_r) \neq 0, \text{ and } \Phi_{u_1 u_2}(\omega_r) \neq 0, \text{ for } r = 3, 6, 9, \dots \qquad (284)$$

where $$\omega_r = \frac{2\pi r}{N_s T}, \quad r = 1, 2, 3, \dots$$

Note that from equation (241) the non-zero cross spectra in (284) assume that

$$\cos(\phi_{p_1} - \phi_{p_2}) \neq 0$$

As we shall see later the optimization problem will have a Weyl constraint that will force the optimizer to select the phases in this manner.

If the number of correlated harmonics in the input is n_c then equation (284) will hold for n_c frequencies. Suppose n is the maximum numerator order of any filter in the transfer function $\Delta G(q)$. As shown earlier in this chapter, we need

1. $\Phi_{u_1} \neq 0$ at at least n frequencies
2. $\Phi_{u_2} \neq 0$ at at least n frequencies

Thus there must be at least n frequencies satisfying (282) and at least n frequencies satisfying (283). Then the persistence of excitation requirement (261) becomes

$$n_s \geq 2n + n_c \qquad (285)$$

In the case studies that we will consider shortly, the above condition will always hold. In addition the correlated harmonics will promote low gain directionality information.

Previous work ([13], [6]) in our research group has dealt with the problem of minimizing the crest factor of input or output signal subject to the move size and bound constraints on both input and output signals. The crest factor of a signal $u(k)$ is defined as the ratio of the ∞-norm to the 2-norm.

$$CF(u) = \frac{||u(k)||_\infty}{||u(k)||_2} \qquad (286)$$

Minimizing the crest factor makes the input power as large as possible while still keeping $u(k)$ bounded by user defined constants, thereby contributing to plant friendliness. The focus will be to excite output from the system in all possible directions in the output state space. The next section describes the Weyl criterion which will be used in developing this new framework for input design.

2.9. Uniform Distribution of Infinite Sequences: The Weyl Criterion

Discrepancy theory deals with the distribution of points in space (see [9]). Weyl criterion [16] gives the necessary and sufficient conditions for a sequence to be uniformly distributed in $[0,\ 1)^d$. Suppose $u = (u_1,\ u_2, \dots)$ be an infinite sequence of points in [0, 1]. The sequence u above is called uniformly distributed in [0, 1] if for each sub-interval $[a,\ b) \in [0,1]$

$$\lim_{n\to\infty}\left(\frac{1}{N}|\,\{u_1, u_2, \dots, u_n\} \cap [a, b)|\right) = b - a \tag{287}$$

Let f: [0, 1] $\to \mathbf{R}$ be a Riemann-integrable function. If the sequence $u = (u_1, u_2, \dots)$ is uniformly distributed in [0, 1] then by definition of uniform distribution it is easy to see that

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} f(u_i)\right) = \int_0^1 f(x)dx \tag{288}$$

Equation (288) is valid for all Riemann-integrable functions. We know that any function $f(x)$ as defined above can be expanded into a complex fourier series as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} c_k e^{2\pi ikx} \tag{289}$$

Here $e^{2\pi ikx}$, $k =$ integer act as basis functions for the expansion. Note that for $k = 0$ the basis function is $e^{2\pi i0x} = 1$ for which equation (288) holds trivially. Hence for $k \neq 0$ the basis functions satisfy equation (288). This means that

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=1}^{n} e^{2\pi iku_j}\right) = \int_0^1 e^{2\pi ikx}dx \tag{290}$$

But
$$\int_0^1 e^{2\pi ikx}dx = \left(\frac{e^{2\pi ikx}}{2\pi ik}\right)_0^1 = 0 \tag{291}$$

From (290) and (291) and the Weirstrass approximation theorem it is easy to prove that a sequence $u = (u_1, u_2, \dots)$ is uniformly distributed in [0, 1] if and only if for all integers $k \neq 0$

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=1}^{n}e^{2\pi ikuj}\right)=0 \tag{292}$$

This result is called the Weyl Criterion. Weyl's criterion can be extended to higher dimensions also.

Theorem 1 (H. Weyl, 1916). *A sequence* $\{y_1(n),\, y_2(n)\}$ is *equidistributed in* $[0,\,1)^2$ *if and only if*

$$\lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(l_1y_1(n)+l_2y_2(n))}=0$$

for all sets of integers l_1, l_2 *not both zero.*

Breaking the above equation into real and imaginary parts we get:

The sequence $\{y_1(n),\, y_2(n)\}$ is equi-distributed in $[0,\,1)^2$ if and only if $\forall$ sets of integers l_1, l_2 not both zero the following conditions hold:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos\left[2\pi(l_1y_1(n)+l_2y_2(n))\right]=0 \tag{293}$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sin\left[2\pi(l_1y_1(n)+l_2y_2(n))\right]=0 \tag{294}$$

2.10. Constrained Problem Formulation

Our goal here, is to design an input signal that has good directionality information in the output state space of the system. This assumes *a priori* knowledge of the plant model either as an equation or a computer program that is available to the optimizer.

We introduce two cycles of input each of length N_s and let the transients die out in the first cycle ($k = 0, \ldots, N_s - 1$) of the output. As before the input $u(k)$ and output $y(k)$ are vectors with two components. To design a plant friendly signal we impose bound constraints on both $u(k)$ and/or $y(k)$ in the second cycle.

$$|y_1(k)| \leq C_{y_1}, k = N_s, \ldots, 2N_s - 1 \tag{295}$$

$$|y_2(k)| \leq C_{y_2}, k = N_s, \ldots, 2N_s - 1 \tag{296}$$

$$|u_1(k)| \leq C_{u_1}, k = N_s, \ldots, 2N_s - 1 \tag{297}$$

$$|u_2(k)| \leq C_{u_2}, k = N_s, \ldots, 2N_s - 1 \tag{298}$$

In the above equations, C_{y_1}, C_{y_1}, C_{u2} and C_{u_2} are user defined constants.

We would also like to have the move size of $u(k)$ and $y(k)$, which is the difference between successive values in $u(k)$ and $y(k)$ to be small. We therefore impose the constraints:

$$|y_1(k+1) - y_1(k)| \leq \Delta C_{y_1}, k = N_s - 1, \ldots, 2N_s - 2 \tag{299}$$

$$|y_2(k+1) - y_2(k)| \leq \Delta C_{y_2}, k = N_s - 1, \ldots, 2N_s - 2 \tag{300}$$

$$|u_1(k+1) - u_1(k)| \leq \Delta C_{u_1}, k = N_s - 1, \ldots, 2N_s - 2 \tag{301}$$

$$|u_2(k+1) - u_2(k)| \leq \Delta C_{u_2}, k = N_s - 1, \ldots, 2N_s - 2 \tag{302}$$

Again $\Delta C_{y_1}, \Delta C_{y_2}, \Delta C_{u_1}$and ΔC_{u_2} are user defined constants.

The a *priori* information about the plant is available as either a model estimated from previous tests or as a computer program simulating the system. These relationships are:

$$y_1(k) = f_1(u_1, u_2, y_1, y_2), \quad k = 0, 2N_s - 1 \tag{303}$$

and

$$y_2(k) = f_2(u_1, u_2, y_1, y_2), \quad k = 0,\ 2N_s - 1 \tag{304}$$

Here the arguments of f_1 and f_2 indicate the dependence of y_1 and y_2 on the values of the vectors u_1, u_2, y_1 and y_2. The inputs $u_1(k)$ and $u_2(k)$ are chosen to be multisine inputs:

$$u_1(k) = \sum_{p=1}^{n_s} \sqrt{2\alpha_{p_1}} \cos\left(\frac{2\pi p}{N_s} k + \phi_{p_1}\right) \tag{305}$$

$$u_2(k) = \sum_{q=1}^{n_s} \sqrt{2\alpha_{q_2}} \cos\left(\frac{2\pi q}{N_s} k\phi_{q_2}\right) \tag{306}$$

We will use the modified zippered spectrum described above:

$$\alpha_{(3j-2)1} \geq 0, \quad \alpha_{(3j-1)1} = 0 \quad \alpha_{(3j)1} \geq 0, \quad j = 1, 2, \ldots \tag{307}$$

and

$$\alpha_{(3j-2)2} = 0, \alpha_{(3j-1)} \geq 0, \alpha_{(3j)2} \geq 0, \quad j = 1, 2, \ldots \tag{308}$$

Our goal is to uniformly distribute the points $(y_1(k),\ y_2(k))$ in the output state space region $[-C_{y_1}, C_{y_1}) \times [-C_{y_2}, C_{y_2})$. We wish to use the Weyl Criterion described in the previous section to achieve this uniform distribution. Since the Weyl Criterion deals with uniform distributions in $[0, 1)^2$, we introduce a change of variables:

$$\hat{y}_1(k) = \frac{y_1(k) + C_{y_1}}{2C_{y_1}} \tag{309}$$

and

$$\hat{y}_2(k) = \frac{y_2(k) + C_{y_2}}{2C_{y_2}} \tag{310}$$

With these change of variables, we can use the Weyl criterion to uniformly distribute $(\hat{y}_1,\ \hat{y}_2)$ in $[0,\ 1)^2$. Since we only have a finite number of points, we cannot impose (293) and (294) as described before.

We choose an integer L and form the set S as follows:

$$S = \{x : x \in Z \,\text{and}\, |x| \leq L\} \tag{311}$$

where Z is the set of all integers. Then define the set W as

$$W = \{(l_1, l_2) : l_1 \in S, l_2 \in S \,\text{and}\, (l_1, l_2) \neq (0, 0)\} \tag{312}$$

We then try to minimize the sum in equations (7) and (8) for all elements of the set W. We impose this "Weyl" constraint on the second cycle ($k = N_s + 1, \ldots, 2N_s - 1$) of the output. The optimization is carried out to estimate the amplitudes and phases $\alpha_{p1}, \alpha_{p2}, \phi_{p1}, \phi_{p2}$ of the multisine input. The complete problem statement is as follows:

$$\min_{\alpha_{p1},\alpha_{p2},\phi_{p1},\phi_{p2}} t \tag{313}$$

$$\forall (l_1, l_2) \in W \text{ s.t.} \sum_{k=N_s+1}^{2N_s-1} \cos\left[2\pi(l_1 \hat{y}_1(k) + l_2 \hat{y}_2(k))\right] \leq t \tag{314}$$

$$\forall (l_1, l_2) \in W \text{ s.t.} \sum_{k=N_s+1}^{2N_s-1} \sin\left[2\pi(l_1 \hat{y}_1(k) + l_2 \hat{y}_2(k))\right] \leq t \tag{315}$$

$$\text{s.t. } |y_1(k)| \leq C_{y_1}, k = N_s, \ldots, 2N_s - 1 \tag{316}$$

$$\text{s.t. } |y_2(k)| \leq C_{y_2}, k = N_s, \ldots, 2N_s - 1 \tag{317}$$

$$\text{s.t. } |u_1(k)| \leq C_{u_1}, k = N_s, \ldots, 2N_s - 1 \tag{318}$$

$$\text{s.t. } |u_2(k)| \leq C_{u_2}, k = N_s, \ldots, 2N_s - 1 \tag{319}$$

$$\text{s.t. } |y_1(k+1) - y_1(k)| \leq \Delta C_{y_1}, k = N_s - 1, \ldots, 2N_s - 2 \tag{320}$$

$$\text{s.t. } |y_2(k+1) - y_2(k)| \leq \Delta C_{y_1}, k = N_s - 1, \ldots, 2N_s - 2 \tag{321}$$

$$\text{s.t. } |u_1(k+1) - u_1(k)| \leq \Delta C_{u_1}, k = N_s - 1, \ldots, 2N_s - 2 \tag{322}$$

$$\text{s.t. } |u_2(k+1) - u_2(k)| \leq \Delta C_{u_2}, k = N_s - 1, \ldots, 2N_s - 2 \tag{323}$$

$$\text{s.t. } u_1(k) = \sum_{p=1}^{n_s} \sqrt{2\alpha_{p1}} \cos\left(\frac{2\pi p}{N_s} k + \phi_{p1}\right) \tag{324}$$

$$\text{s.t. } u_2(k) = \sum_{q=1}^{n_s} \sqrt{2\alpha_{q2}} \cos\left(\frac{2\pi q}{N_s} k + \phi_{q2}\right) \tag{325}$$

$$\text{s.t. } \alpha_{(3j-2)1} \geq 0, \alpha_{(3j-1)1} = 0, \alpha_{(3j)1} \geq 0, \ j = 1, 2, \ldots \tag{326}$$

$$\text{s.t. } \alpha_{(3j-2)2} \geq 0, \alpha_{(3j-1)2} = 0, \alpha_{(3j)2} \geq 0, \ j = 1, 2, \ldots \tag{327}$$

$$\text{s.t. } y_1(k) = f_1(u_1, u_2, y_1, y_2), k = 0, 2N_s - 1 \tag{328}$$

$$\text{s.t. } y_2(k) = f_2(u_1, u_2, y_1, y_2), k = 0, 2N_s - 1 \quad (329)$$

$$\text{s.t. } \hat{y}_1(k) = \frac{y_1(k) + C_{y_1}}{2C_{y_1}} \quad (330)$$

$$\text{s.t. } \hat{y}_2(k) = \frac{y_2(k) + C_{y_2}}{2C_{y_2}} \quad (331)$$

$$\text{s.t. } t \ \geq \ \varepsilon \quad (332)$$

The lower bound constraint on t is imposed to promote faster convergence. ε is chosen to be some small positive constant. This describes the complete set of constraints that we will use in the 3 case study to follow.

2.11. Note on the Computer Solution

The constrained problems were solved by programming them in the modelling language AMPL which has built in automatic differentiation up to second order derivatives. The Weyl constraints are twice continuously differentiable and so the optimizer can make direct use of second derivative information. The optimizer used was KNITRO developed by Byrd and co-workers [2], [1]. KNITRO is an interior point trust region SQP solver and is suitable for solving both large and small problems.

The 3 Case Study: Nonlinear High-Purity Distillation Process

Multivariable dynamical systems, in contrast to single variables may be ill-conditioned and highly interactive; this phenomena presents challenges for both system identification and subsequent controller design. A demanding nonlinear and highly interactive multivariable process system that benefits from judiciously applied system identification techniques is high purity distillation (Fig. 1); the methanol-ethanol distillation column model developed by [8] is commonly used as a benchmark problem [3, 15]. In a typical binary high-purity distillation column (such as the Weischedel-McAvoy column), the objective is to separate a two-component mixture into streams that are very pure in one component in each of the product streams (y_D in the distillate D and x_B in the bottom B stream). Reflux (L) and boilup (V) flows can be used as manipulated variables to maintain the column at desired operation. The highly interactive nature of high-purity distillation is reflected in the fact that dynamically the system will tend to respond in the principal gain direction (consisting of achieving greater purity in one stream at the expense of purity in the other) while the low gain direction (reflecting conditions where purities in both the distillate and bottom streams increase simultaneously) is much less evident. An illustration of this phenomenon for the column model per [8] using a standard multisine signal design [6, 13] is shown Fig. 2.

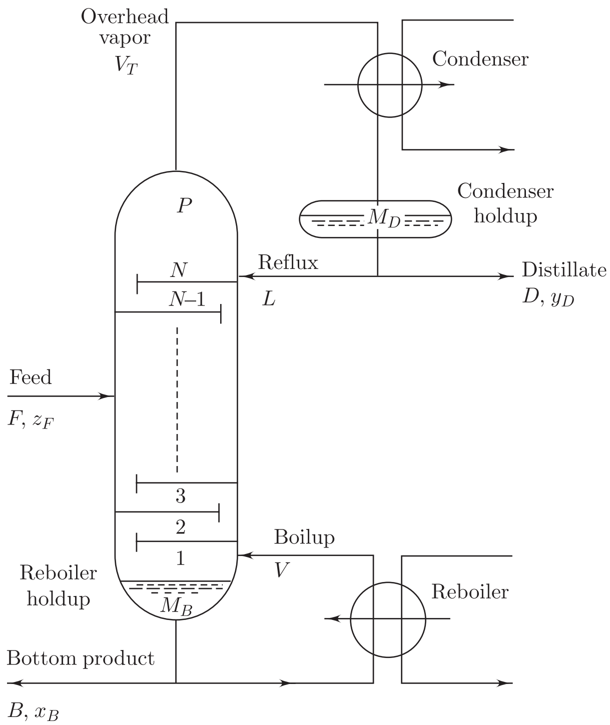
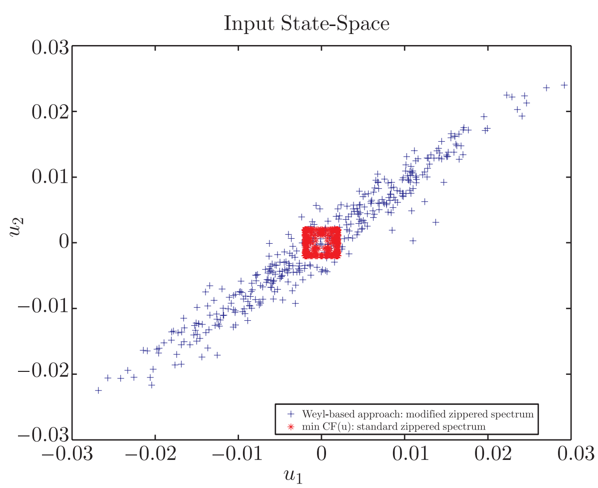
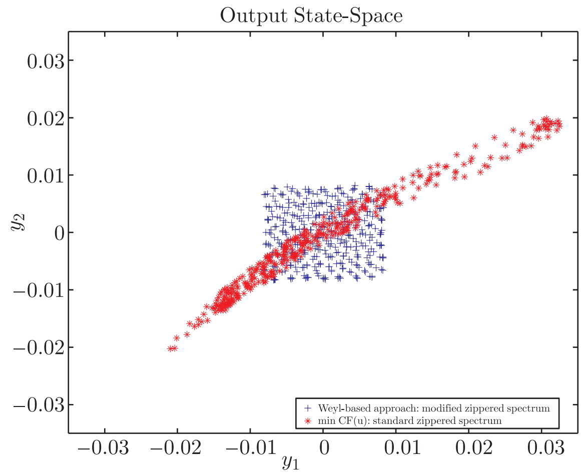


Fig. 1. Binary distillation column schematic, per Morari and Zafiriou (1988).



(a) Input State-space

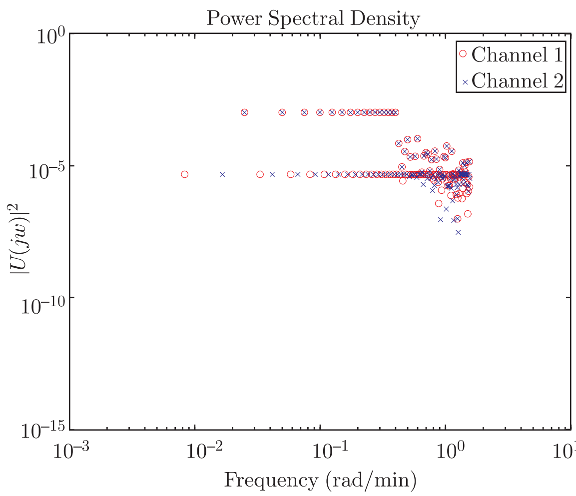


(b) Output State-space

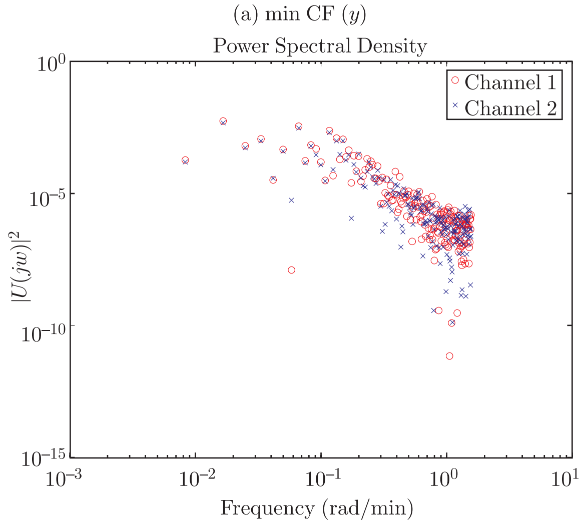
Fig. 2. Comparison of the input and output state-spaces for a standard "zippered" multisine (*) vs. a Weyl-based signal design with modified spectrum (+) for the Weischedel-McAvoy distillation column.

To address the demands of highly interactive systems, one approach is to modify the standard multisine signal to contain correlated harmonics with high levels of power, which improve the low gain-direction content in the data and promote better coverage of the output state-space. The work in [6] and [13] present a design procedure where a crest factor objective function is minimized in either u or y (min CF(w) or min CF(y)). Design parameters determined on the basis of the guidelines per [6] and [13] using dominant time constant estimates ($\tau_{dom}^{L} = 5$ and $\tau_{dom}^{H} = 20$ min) and user choices of $\delta = 0, \alpha_s = 2$, and $\beta_s = 3$, lead to parameter settings of $T = 2$ minutes, $n_s = 189$, and $N_s = 378$. A value of the amplification factor $\gamma = 15$ was chosen for the signal with modified spectrum; the resulting input spectrum for this signal is shown in Fig. 3(a). Input and output state-space plots are shown in Figs. 2 and 4. More detailed results are given in [12].

A significant benefit of the optimization-based problem formulation presented is that nonlinear model forms can be readiliy incorporated in the design procedure, which results in an improved ability to both meet plant-friendliness requirements as well as address the directionality and uniform distribution requirements in the output for demanding applications. A polynomial Nonlinear AutoRegressive with eXternal (NARX) input model



(a) min CF (y)

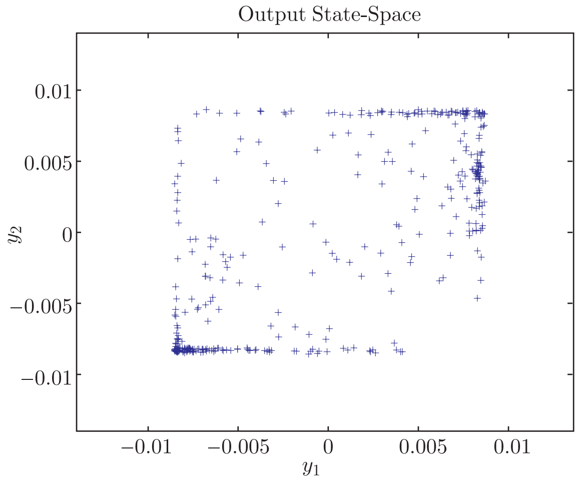


(b) Weyl-based Approach

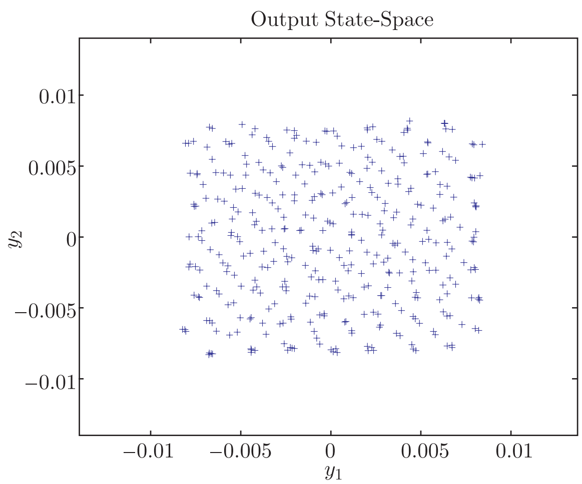
Fig. 3. Input power spectral densities for Weischedel-McAvoy distillation column: min CF(y) modified zippered spectrum signal (a) versus Weyl-based design (b).

with structure as proposed by [15]

$$
\begin{aligned}
y(k) = \theta^{(0)} + \sum_{i=1}^{n_y} \theta_i^{(1)} y(k-i) + \sum_{i=p}^{n_u} \theta_i^{(2)} u(k-i) + \sum_{i=1}^{n_y} \sum_{j=1}^{i} \theta_{(i,j)}^{(3)} y(k-i) y(k-j) \\
+ \sum_{i=\rho}^{n_u} \sum_{j=\rho}^{i} \theta_{(i,j)}^{(4)} u(k-i) u(k-j) \qquad (333) \\
+ \sum_{i=1}^{n_y} \sum_{j=\rho}^{n_u} \theta_{(i,j)}^{(5)} y(k-i) u(k-j) + \ldots
\end{aligned}
$$



(a) min CF (y)



(b) Weyl-based Approach

Fig. 4. Output state-space analysis for Weischedel-McAvoy distillation column: min CF(y) modified zippered spectrum signal (a) versus Veyl-based design (b).

was estimated for the Weischedel-McAvoy column and used to generate output predictions for the optimizer in both the min CF(y) and Weyl-based signals design scenarios. The benefits of the Weyl-based formulation over the minimum crest factor signal design in producing a uniform distribution in the output state-space of the data can be clearly seen by contrasting Fig. 4 (a and b): the use of the Weyl-based criterion clearly results in a much more uniformly distributed coverage of the state-space, and a much better dataset for data-centric estimation purposes. The uniform distribution of the output within the bounds specified in the problem results in a natural balance between the high and low gain information content in the data.

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