

The parabolic control problem is given by  
minimize

$$f(y, u) = \frac{1}{2} \int_0^l (y(x, T) - y_T(x))^2 dx + \frac{\alpha}{2} \int_0^T u(t)^2 dt \\ + \int_0^T (a_y(t)y(l, t) + a_u(t)u(t)) dt, \quad \alpha > 0$$

subject to

$$y_t - y_{xx} = 0 \quad \text{in } (0, l) \times (0, T) \\ y(x, 0) = a(x) \quad \text{in } (0, l) \\ y_x(0, t) = 0 \quad \text{in } (0, T) \\ y_x(l, t) + \beta y(l, t) = b(t) + u(t) - \varphi(y(l, t)) \quad \text{in } (0, T) \\ \alpha_1 \leq u(t) \leq \alpha_2 \quad \text{in } (0, T) \\ \alpha_3 \leq y(x, t) \leq \alpha_4 \quad \text{in } (0, l) \times (0, T)$$

As problem 5.1 this is solved without state constraints and with the following data

$$l = \pi/4, \quad T = 1, \quad \alpha = \frac{\sqrt{2}}{2}(e^{2/3} - e^{1/3}) \\ y_T(x) = (e + e^{-1}) \cos x, \quad \alpha_1 = 0, \quad \alpha_2 = 1 \\ a(x) = \cos x, \quad a_y(t) = -e^{-2t}, \quad a_u(t) = \frac{\sqrt{2}}{2}e^{1/3} \\ b(t) = \frac{1}{4}e^{-4t} - \min \left( 1, \max \left( 0, \frac{e^t - e^{1/3}}{e^{2/3} - e^{1/3}} \right) \right) \\ \varphi(y) = y|y|^3, \quad \beta = 1. \tag{1}$$

A local optimum for this problem is the pair  $(\bar{y}, \bar{u})$ ,

$$\bar{y}(x, t) = e^{-t} \cos x \\ \bar{u}(t) = \min \left( 1, \max \left( 0, \frac{e^t - e^{1/3}}{e^{2/3} - e^{1/3}} \right) \right).$$

As example 5.2 the above control problem is solved with the data

$$\begin{aligned}
 & l = 1, \quad T = 1.58, \quad \alpha = .001 \\
 & y_T(x) = .5(1 - x^2), \\
 & \alpha_1 = -1, \quad \alpha_2 = 1 \quad \alpha_3 = 0, \quad \alpha_4 = 1 \\
 & a(x) = 0, \quad a_y(t) = 0, \quad a_u(t) = 0, \\
 \text{(I)} \quad & b(t) = 0, \quad \varphi(y) = 0, \quad \beta = 1 \\
 \text{(II)} \quad & b(t) = 0, \quad \varphi(y) = y^2, \quad \beta = 0 \\
 \text{(III) as (II) and} \quad & \alpha_3 = 0, \quad \alpha_4 = .675
 \end{aligned} \tag{2}$$

Example 5.2.IV is the instationary Burgers equation

$$y_t = \nu y_{xx} - yy_x \quad \text{in } (0, l) \times (0, T)$$

with same data as for case (II) above with  $\nu = .01$  and the control bounds  $\alpha_1 = 0.1, \alpha_2 = 0.6$  both of which become active in the solution.

Problem 5.400 is the same as problem 5.2-I except for  $\alpha = .05$  and the state constraints  $\alpha_3 = -1, \alpha_4 = 1$

Problem (P) is given by

Minimize

$$\begin{aligned}
 J(y, u) = & \frac{1}{2} \int \int_Q \alpha(x, t) (y(x, t) - y_d(x, t))^2 dx dt + \frac{\nu}{2} \int_0^T u^2(t) dt \\
 & + \int_0^T [a_y(t) y(l, t) + a_u(t) u(t)] dt
 \end{aligned} \tag{3}$$

subject to

$$\begin{aligned}
 y_t - y_{xx} &= e_Q && \text{in } Q \\
 y(x, 0) &= 0 && \text{in } (0, l) \\
 y_x(0, t) &= 0 && \text{in } (0, T) \\
 y_x(l, t) + y^2(l, t) &= e_\Sigma(t) + u(t) && \text{in } (0, T)
 \end{aligned} \tag{4}$$

and to

$$u_a \leq u(t) \leq u_b, \quad \text{a.e. in } (0, T), \quad (5)$$

$$\int \int_Q y(x, t) \, dxdt \leq 0. \quad (6)$$

In this setting,  $T, \nu, l > 0, u_a < u_b$  are fixed real numbers,  $Q = (0, l) \times (0, T)$ . Functions  $\alpha, y_d$ , and  $e_Q$  are given in  $L^\infty(Q)$ , and  $a_y, a_u, e_\Sigma$  are fixed in  $L^\infty(0, T)$ . We shall denote the set of admissible controls by  $U_{ad} = \{u \in L^\infty(0, T), | u_a \leq u \leq u_b, \quad \text{a.e. } \in (0, T)\}$ .

Problem (P) is nonconvex, since the state equation is semilinear. Its nonlinearity  $y^2$  is not of monotone type, hence standard results on existence and uniqueness of solutions to (4) do not apply.

We fix here the following quantities in (P):

$$T = 1, \quad l = \pi, \quad u_a = 0, \quad u_b = 1, \quad \nu = 0.004$$

$$\alpha(x, t) = \begin{cases} \alpha_o \in \mathbb{R}, & t \in [0, 1/4] \\ 1, & t \in (1/4, 1], \end{cases}$$

$$y_d(x, t) = \begin{cases} \frac{1}{\alpha(x, t)}(1 - (2 - t)\cos x), & t \in [0, 1/2] \\ \frac{1}{\alpha(x, t)}(1 - (2 - t - \alpha(x, t)(t - 1/2)^2))\cos x, & t \in (1/2, 1], \end{cases}$$

$$a_y(t) = \begin{cases} 0, & t \in [0, 1/2] \\ 2(t - 1/2)^2(1 - t), & t \in (1/2, 1], \end{cases}$$

$$a_u(t) = \nu + 1 - (1 + 2\nu)t,$$

$$e_Q(t) = \begin{cases} 0, & t \in [0, 1/2] \\ (t^2 + t - 3/4)\cos x, & t \in (1/2, 1], \end{cases}$$

$$e_\Sigma(t) = \begin{cases} 0, & t \in [0, 1/2] \\ (t - 1/2)^4 - (2t - 1), & t \in (1/2, 1]. \end{cases}$$

The quantities

$$\begin{aligned}\bar{u} &= \max\{0, 2t - 1\} \\ \bar{y} &= \begin{cases} 0, & t \in [0, 1/2] \\ (t - 1/2)^2 \cos x, & t \in (1/2, 1] \end{cases} \\ \bar{p} &= (1 - t) \cos x \\ \bar{\lambda} &= 1\end{aligned}$$

satisfy the system of first order necessary conditions.

For problem TWOD see the top of the AMPL model.