A boundary control problem

This example is meant to be typical for practical problems that have to be solved in industrial and other applications. A mathematical description of the problem is as follows. The underlying boundary value problem is Laplace's equation on the unit square, corresponding to no internal heat sources, coupled with mixed boundary conditions, namely homogeneous Neumann conditions on $x_2 = 0$, or no heat flux across this boundar y, a heat flux proportional to the temperature at the boundaries $x_1 = 0$ and $x_1 = 1$, while the solution is controlled on $x_2 = 1$. The control function is to be found such that the temperature in the central subsquare of length 0.5 is as close as possible to a given function $y_d = 1$ in the L_2–norm. In the first version of the problem a multiple α of a regularizing boundary integral over the control function is added to the objective functional, while without this a bang-bang control may be expected in the second version. To complete the problem definition upper and lower bounds of 10 respectively 0 are imposed on both state and control.

Thus letting $\Gamma_2 = \{\,(x_1, 1) \mid 0 \leq x_1 \leq 1\,\}$ and $\Omega_0 = [0.25, 0.75]^2$, the control problem is to determine a function $u \in L^\infty(\Gamma_2)$ which minimizes

$$F(y,u) = \frac{1}{2}\int_{\Omega_0} (y(x)-1)^2\,dx + \frac{\alpha}{2}\int_{\Gamma_2} u(x)^2\,dx \tag{1}$$

subject to the state equation, Neumann and Dirichlet boundary conditions and control and state inequality constraints,

$$\begin{array}{rclllll}
-\Delta y(x) & = & 0 & \text{in} & \Omega\,, & & \\
\partial_\nu y(x) & = & 0 & \text{for} & x_2 = 0, & 0 \leq x_1 \leq 1\,, & \\
\partial_\nu y(x) & = & y(x) - 5 & \text{for} & x_1 \in \{0,1\}, & 0 \leq x_2 \leq 1\,, & \\
y(x) & = & u(x) & \text{for} & x_2 = 1, & 0 \leq x_1 \leq 1\,, & (2) \\
y(x) & \leq & 3.15 & \text{in} & \Omega_0\,, & & \\
y(x) & \leq & 10 & \text{in} & \Omega \setminus \Omega_0\,, & & \\
0 \leq u(x) & \leq & 10 & \text{for} & x_2 = 1, & 0 \leq x_1 \leq 1\,. &
\end{array}$$

Bdry 1 $\alpha = .005$

Bdry 2 $\alpha = 0$

A distributed control example

The problem is to determine a distributed control $u \in L^\infty(\Omega)$ that minimizes the functional

$$F(y,u) = \int_\Omega (Mu(x)^2 - Ku(x)y(x))\,dx \tag{3}$$

subject to the elliptic state equation

$$-\Delta y(x) = y(x)(a(x) - u(x) - by(x))\,, \quad \text{for} \quad x \in \Omega\,, \tag{4}$$

homogeneous Neumann boundary conditions,

$$\partial_\nu y(x) = 0\,, \qquad \text{for} \quad x \in \Gamma\,, \tag{5}$$

and control and state inequality constraints

$$u_1 \le u(x) \le u_2 \quad y(x) \le \psi(x)\,, \quad \text{for} \quad x \in \Omega\,. \tag{6}$$

The following concrete data were used:

Dist 1

$a(x) = 7 + 4\sin(2\pi x_1 x_2)$, $b = 1$, $M = 1$, $K = 0.8$,
$u_1 = 1.7$, $u_2 = 2$, $\psi(x) = 7.1$.

Dist 2

$a(x) = 7 + 4\sin(2\pi x_1 x_2)$, $b = 1$, $M = 0$, $K = 0.8$,
$u_1 = 2$, $u_2 = 6$, $\psi(x) = 4.8$.