Dirichlet problems

The problem is to determine a control $u \in L^\infty(\Gamma)$ that minimizes the functional

$$F(y,u) = \frac{1}{2}\int\limits_\Omega (y(x) - y_d(x))^2\,dx + \frac{\alpha}{2}\int\limits_\Gamma (u(x) - u_d(x))^2\,dx \qquad (1)$$

with given functions $y_d \in C(\bar\Omega)$, $u_d \in L^\infty(\Gamma)$, and a nonnegative weight $\alpha \geq 0$. subject to the state equation

$$\begin{aligned} -\Delta y(x) + d(x, y(x)) &= 0\,, && \text{for} \quad x \in \Omega\,, \\ y(x) &= b(x, u(x))\,, && \text{for} \quad x \in \Gamma\,, \end{aligned} \qquad (2)$$

and the inequality constraints on control and state

$$y(x) \leq \psi(x) \quad \text{on } \Omega\,, \qquad u_1(x) \leq u(x) \leq u_2(x) \quad \text{on } \Gamma\,, \qquad (3)$$

with functions $\psi \in C(\bar\Omega)$ and $u_1, u_2 \in L^\infty(\Gamma)$. In this setting, $\Omega \subset I\!\!R^2$ is a bounded domain with piecewise smooth boundary $\Gamma = \partial\bar\Omega$. Note that the state inequality constraints (3) are supposed to hold on the closure of Ω.

Example 1:

on Ω : $-\Delta y(x) = 20$, $y(x) \leq 3.5$, $y_d(x) = 3 + 5x_1(x_1 - 1)x_2(x_2 - 1)$,
on Γ : $y(x) = u(x)$, $0 \leq u(x) \leq 10$, $u_d(x) \equiv 0$, $\alpha = 0.01$.

Example 2:

on Ω : $-\Delta y(x) = 20$, $y(x) \leq 3.5$, $y_d(x) = 3 + 5x_1(x_1 - 1)x_2(x_2 - 1)$,
on Γ : $y(x) = u(x)$, $0 \leq u(x) \leq 10$, $u_d(x) \equiv 0$, $\alpha = 0$.

Example 3:

on Ω : $-\Delta y(x) = 20$, $y(x) \leq 3.2$, $y_d(x) = 3 + 5x_1(x_1 - 1)x_2(x_2 - 1)$,
on Γ : $y(x) = u(x)$, $1.6 \leq u(x) \leq 2.3$, $u_d(x) \equiv 0$, $\alpha = 0.01$.

Example 4:

on Ω : $-\Delta y(x) = 20$, $y(x) \leq 3.2$, $y_d(x) = 3 + 5x_1(x_1 - 1)x_2(x_2 - 1)$,
on Γ : $y(x) = u(x)$, $1.6 \leq u(x) \leq 2.3$, $u_d(x) \equiv 0$, $\alpha = 0$.

Neumann problems

The problem is to determine a control $u \in L^\infty(\Gamma)$ that minimizes the functional

$$F(y,u) = \frac{1}{2} \int_\Omega (y(x) - y_d(x))^2 \, dx + \frac{\alpha}{2} \int_\Gamma (u(x) - u_d(x))^2 \, dx \tag{4}$$

with given functions $y_d \in C(\bar\Omega)$, $u_d \in L^\infty(\Gamma)$, and a nonnegative weight $\alpha \geq 0$. subject to the state equation

$$\begin{aligned} -\Delta y(x) + d(x, y(x)) &= 0\,, && \text{for} \quad x \in \Omega\,, \\ \partial_\nu y(x) &= b(x, y(x), u(x))\,, && \text{for} \quad x \in \Gamma\,, \end{aligned} \tag{5}$$

and the inequality constraints on control and state

$$y(x) \leq \psi(x) \quad \text{on } \Omega\,, \qquad u_1(x) \leq u(x) \leq u_2(x) \quad \text{on } \Gamma\,, \tag{6}$$

with functions $\psi \in C(\bar\Omega)$ and $u_1, u_2 \in L^\infty(\Gamma)$. In this setting, $\Omega \subset I\!R^2$ is a bounded domain with piecewise smooth boundary $\Gamma = \partial\,\bar\Omega$. The derivative in the direction of the outward unit normal ν of Γ is denoted by ∂_ν in (2). Note that the state inequality constraints (3) are supposed to hold on the closure of Ω.

Example 5:

on Ω : $-\Delta y(x) = 0,$ $\quad y(x) \leq 2.071,$
$y_d(x) = 2 - 2(x_1(x_1 - 1) + x_2(x_2 - 1)),$

on Γ : $\partial_\nu \, y(x) = u(x) - y(x)^2,$ $\quad 3.7 \leq u(x) \leq 4.5,\ u_d(x) \equiv 0\,,\ \alpha = 0.01.$

Example 6:

on Ω : $-\Delta y(x) = 0,$ $\quad y(x) \leq 2.835,$
$y_d(x) = 2 - 2(x_1(x_1 - 1) + x_2(x_2 - 1)),$

on Γ : $\partial_\nu \, y(x) = u(x) - y(x)^2,$ $\quad 6 \leq u(x) \leq 9,\ u_d(x) \equiv 0\,,\ \alpha = 0.$

Example 7:

on Ω : $-\Delta y(x) - y(x) + y(x)^3 = 0,$ $\quad y(x) \leq 2.7,$
$y_d(x) = 2 - 2(x_1(x_1 - 1) + x_2(x_2 - 1)),$

on Γ : $\partial_\nu \, y(x) = u(x),$ $\quad 1.8 \leq u(x) \leq 2.5,\ u_d(x) \equiv 0\,,\ \alpha = 0.01.$

Example 8:

on Ω : $-\Delta y(x) - y(x) + y(x)^3 = 0,$ $\quad y(x) \leq 2.7,$
$y_d(x) = 2 - 2(x_1(x_1 - 1) + x_2(x_2 - 1)),$

on Γ : $\partial_\nu \, y(x) = u(x),$ $\quad 1.8 \leq u(x) \leq 2.5,\ u_d(x) \equiv 0\,,\ \alpha = 0.$