Generalized sampling
A new framework for image and signal reconstruction

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Outline of the talk

Introduction

Generalized sampling

Reconstructions from Fourier samples

Generalized sampling for nonuniform samples

Generalized sampling and infinite-dimensional compressed sensing
Introduction

Generalized sampling

Reconstructions from Fourier samples

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Generalized sampling and infinite-dimensional compressed sensing
Reconstructions from the Fourier transform

Fundamental problem: recover a image/signal $f$ from pointwise samples of its Fourier transform (FT)

$$\mathcal{F} f(\omega) = \int_{\mathbb{R}^d} f(x) e^{2i\pi \omega \cdot x} \, dx.$$ 

E.g. Magnetic Resonance Imaging (MRI).
Key issues

1. The sampling scheme is **fixed** and cannot be altered.

2. Taking many samples is expensive/infeasible. Thus, one wants to reconstruct $f$ using **as few samples** as possible.

3. Sampling frequencies $\{\omega_n\}$ may be **uniform** or **nonuniform**.

4. Samples are always **noisy**. Also other effects, e.g. **jitter**.
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The Shannon sampling theorem

Let $f \in L^2(\mathbb{R})$ with $\text{supp}(f) \subseteq [-1, 1]$. Then $f$ is determined uniquely by

$$\{ \mathcal{F}f(n\epsilon) \}_{n \in \mathbb{Z}}, \quad (\epsilon \leq \frac{1}{2}).$$

Specifically,

$$f(x) = \epsilon \sum_{n \in \mathbb{Z}} \mathcal{F}f(n\epsilon)e^{2\pi i \epsilon nx}.$$  

However, in practice, we cannot access or process all samples of $f$. Thus, Shannon’s Theorem gives rise to the approximation

$$f_N(x) = \epsilon \sum_{n=-N}^{N} \mathcal{F}f(n\epsilon)e^{2\pi i \epsilon nx}.$$  

- In other words, the partial Fourier series of $f$.  

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Fourier series reconstructions

Let $\epsilon = \frac{1}{2}$, $N = 50$:

The function $f(x)$
Fourier series reconstructions

Let $\epsilon = \frac{1}{2}$, $N = 50$:

The functions $f(x)$ and $f_N(x)$
Fourier series reconstructions

Let $\epsilon = \frac{1}{2}$, $N = 50$:

The error $f(x) - f_N(x)$
Fourier series reconstructions

Let $\epsilon = \frac{1}{2}$, $N = 100$:

The error $f(x) - f_N(x)$
Fourier series reconstructions

Let $\epsilon = \frac{1}{2}$, $N = 200$:

The error $f(x) - f_N(x)$

The coefficients $\mathcal{F}f(n\epsilon)$ decay very slowly as $|n| \to \infty$. 
Main question

Given the first $N$ Fourier samples $\{\mathcal{F}f(n\epsilon)\}_{n=-N}^{N}$, is there a better way to recover $f$ than the Fourier series $f_N$?
Other ways to reconstruct $f$ from its samples

Both functions very poorly represented by $f_N$. However,

- $f_1$ is very well approximated by piecewise polynomials,
- $f_2$ is very well approximated by Haar wavelets.
Other ways to reconstruct \( f \) from its samples

Write

\[
    f_i(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n, \quad i = 1, 2,
\]

where \( \{\phi_n\}_{n \in \mathbb{N}} \) are either piecewise polynomials or Haar wavelets.

\[
\begin{array}{c}
\text{coefficients} \ \{\alpha_n\} \ \text{for} \ f_1 \\
\text{coefficients} \ \{\alpha_n\} \ \text{for} \ f_2
\end{array}
\]

- In either case, if we knew \( \alpha_1, \ldots, \alpha_{32} \) we could recover \( f_i \) with error \( \approx 10^{-10} \).
Main problem

More generally, let

$$\hat{f}_n, \quad n \in \mathbb{N},$$

be measurements of $f$ (e.g. samples of $\mathcal{F}f$).

Key assumption: suppose that we know that $f$ has a ‘good’
representation in a basis $\{\phi_n\}$, i.e. $\alpha_n \to 0$ rapidly.

Main problem:

Given the first $N$ measurements $\{\hat{f}_n\}_{n=1}^N$ recover the coefficients
$\{\alpha_n\}$ in the basis $\{\phi_n\}$. 
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More generally, let
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Key assumption: suppose that we know that \( f \) has a ‘good’ representation in a basis \( \{ \phi_n \} \), i.e. \( \alpha_n \to 0 \) rapidly.

Main problem:

Given the first \( N \) measurements \( \{ \hat{f}_n \}_{n=1}^{N} \) recover the coefficients \( \{ \alpha_n \} \) in the basis \( \{ \phi_n \} \).
Key consideration I: quasi-optimality

Suppose that we have a mapping

\[ \mathcal{L} : \{ \hat{f}_1, \ldots, \hat{f}_N \} \mapsto \{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_M \}. \]

The coefficients \( \alpha_1, \alpha_2, \ldots \).
Key consideration I: quasi-optimality

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The coefficients \( \alpha_1, \alpha_2, \ldots \) and approximate coefficients \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_M \).
Key consideration I: quasi-optimality

Suppose that we have a mapping

$$\mathcal{L} : \{\hat{f}_1, \ldots, \hat{f}_N\} \mapsto \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M\}.$$ 

Error equation:

$$f - \sum_{n=1}^M \tilde{\alpha}_n \phi_n = \sum_{n=1}^M (\alpha_n - \tilde{\alpha}_n) \phi_n + \sum_{n=M+1}^{\infty} \alpha_n \phi_n$$

total error  \quad \text{regularization error} \quad \text{truncation error}

It's important that

regularization error \approx \text{truncation error}, \quad \text{(quasi-optimality).}$$
Key consideration II: numerical stability

The mapping \( \mathcal{L} : \{\hat{f}_1, \ldots, \hat{f}_N\} \mapsto \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M\} \) should be numerically stable, i.e. the condition number

\[
\|\mathcal{L}\|\|\mathcal{L}^{-1}\| \ll \infty,
\]

to avoid large errors due to

- round-off error,
- noise,
- jitter,
- shock capturing.
Introduction

Generalized sampling

Reconstructions from Fourier samples

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Generalized sampling and infinite-dimensional compressed sensing
Hilbert space formulation

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.  

- Let $\{\psi_n\}_{n \in \mathbb{N}}$ be an orthonormal sampling basis.  
- Let $\{\phi_n\}_{n \in \mathbb{N}}$ be an orthonormal reconstruction basis.

E.g. Fourier sampling: $\mathcal{H} = L^2(-1, 1)$, $\psi_n(x) := e^{2\pi i \epsilon nx}$.

The reconstruction problem

Given the first $N$ measurements

$$\hat{f}_n = \langle f, \psi_n \rangle, \quad n = 1, \ldots, N,$$

of $f \in \mathcal{H}$, compute the coefficients $\{\alpha_n\}_{n \in \mathbb{N}}$ of $f$ with respect to the reconstruction basis $\{\phi_n\}_{n \in \mathbb{N}}$.  

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Key idea

Allow the parameters

- \( N \) – the number of measurements,
- \( M \) – the number of coefficients \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_M \) to be computed, to differ. Specifically, let \( N > M \).
Best possible reconstruction

The best reconstruction of $M$ coefficients is obviously

$$\tilde{\alpha}_m = \alpha_m = \langle f, \phi_m \rangle, \quad m = 1, \ldots, M.$$ 

The reconstruction

$$f_M = \sum_{m=1}^{M} \alpha_m \phi_m,$$

is the **orthogonal projection** of $f$ onto

$$T_M = \text{span}\{\phi_1, \ldots, \phi_M\} \subset H, \quad \text{(reconstruction space)}.$$ 

Of course, we don’t know $\{\alpha_m\}_{m=1}^{M}$. However, note that, by definition,

$$\langle f_M, \phi_m \rangle = \langle f, \phi_m \rangle, \quad m = 1, \ldots, M.$$
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Generalized sampling

Define $\mathcal{P}_N : H \to S_N := \text{span}\{\psi_1, \ldots, \psi_N\}$ by

$$\mathcal{P}_Ng = \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n.$$  

Note: $\mathcal{P}_N$ is the orthogonal projection onto $S_N$.

Generalized sampling: define $f_{N,M} = \sum_{m=1}^{M} \tilde{\alpha}_m \phi_m \in T_M$ by

$$\langle \mathcal{P}_N f_{N,M}, \phi_m \rangle = \langle \mathcal{P}_N f, \phi_m \rangle, \quad n = 1, \ldots, M.$$  

- A linear system for $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M$ involving only the given measurements $\hat{f}_1, \ldots, \hat{f}_N$.  

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- A linear system for $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M$ involving only the given measurements $\hat{f}_1, \ldots, \hat{f}_N$. 

Recall that

\[ \langle f_{M}, \phi_{m} \rangle = \langle f, \phi_{m} \rangle, \quad m = 1, \ldots, M, \]  

(1)

and

\[ \langle P_{N}f_{N,M}, \phi_{m} \rangle = \langle P_{N}f, \phi_{m} \rangle, \quad n = 1, \ldots, M. \]  

(2)

The operators \( P_{N} \to I \) strongly on \( \mathbb{H} \) as \( N \to \infty \). Thus

\[ f_{N,M} \approx f_{M}, \quad N \to \infty. \]

Hence for sufficiently large \( N \), we expect ‘good’ behaviour of \( f_{N,M} \).
Main theorem

Let

\[ C_{N,M} = \inf \{ \| P_N \phi \| : \phi \in T_M, \| \phi \| = 1 \} \].

Key point: for fixed \( M \), \( C_{N,M} \to 1 \) as \( N \to \infty \).

Theorem (BA, Hansen)

For each \( M \in \mathbb{N} \), there exists an \( N_0 \in \mathbb{N} \) such that \( f_{N,M} \) exists and is unique for all \( N \geq N_0 \), and satisfies the sharp bounds

\[ \| f - f_M \| \leq \| f - f_{N,M} \| \leq \frac{1}{C_{N,M}} \| f - f_M \|. \]

Specifically, \( N_0 \) is the least \( N \) such that \( C_{N,M} > 0 \).
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Geometric interpretation

The map $f \mapsto f_{N,M}$ is precisely the oblique projection onto $T_M$ along $[\mathcal{P}_N(T_M)]^\perp$. Moreover,

$$C_{N,M} = \cos \theta,$$

where $\theta$ is the angle between the subspaces $T_M$ and $\mathcal{P}_N(T_M)$.

$\mathcal{P}_N(T_M)$ and $[\mathcal{P}_N(T_M)]^\perp$ cannot be near-perpendicular for large $N$. Hence $f_{N,M}$ is well-defined, and $f_{N,M} \approx f_M$. 
Numerical implementation

The equations

\[ \langle \mathcal{P}_N f_{N,M}, \phi_m \rangle = \langle \mathcal{P}_N f, \phi_m \rangle, \quad n = 1, \ldots, M, \]

are equivalent to a \( N \times M \) linear least squares system for \( \{\tilde{\alpha}_m\}_{m=1}^M \).

- One can also show that the condition number

\[ \| \mathcal{L} \| \| \mathcal{L}^{-1} \| \leq \frac{1}{C_{N,M}}, \]

where \( \mathcal{L} : \{\hat{f}_1, \ldots, \hat{f}_N\} \mapsto \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M\} \).

- The total computational cost in computing \( f_{N,M} \) is at worst

\[ O \left( \frac{1}{C_{N,M}} NM \right). \]
The stable sampling rate

Define the stable sampling rate

$$\Theta(M; \theta) = \min \left\{ N \in \mathbb{N} : C_{N,M} > \theta \right\}, \quad \theta \in (0, 1).$$

For given $M$, setting $N \geq \Theta(M; \theta)$ ensures

1. Existence and uniqueness of $f_{N,M}$.
2. Numerical stability: $\|L\|\|L^{-1}\| \leq \frac{1}{\theta}$.
3. Quasi-optimality: $\|f - f_{N,M}\| \leq \frac{1}{\theta} \|f - f_M\|$.

Note:

- This is a fundamentally new viewpoint on reconstruction.
- $\Theta(M; \theta)$ is a realization of certain concepts in computational spectral theory concerning the computation of spectra of operators.
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Behaviour of the stable sampling rate

\( \Theta(M; \theta) \) can always be **computed numerically**. However, it is also vitally important to determine **analytic bounds**.

Examples of known bounds:

1. Fourier samples, Haar wavelets: \( \Theta(M; \theta) = c_\theta M \).
2. Fourier samples, (piecewise) polynomials: \( \Theta(M; \theta) = c_\theta M^2 \).
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Generalized sampling and infinite-dimensional compressed sensing
Problem formulation

We seek to reconstruct $f$ in terms of piecewise orthogonal polynomials on $[-1, 1]$.

**Note:** in practice one needs to locate $x_1, \ldots, x_l$ to high accuracy.

- Known as **edge detection**, e.g. concentration kernel methods (Gelb, Tadmor, Tanner,...).
- Edge detection is an important **source of errors**. Any method must be **robust** w.r.t. such errors.
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Numerical example I

Left: $f(x) = \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}(x) \sin(\cos x)$. Right: Fourier series (black), generalized sampling with $N = 25$, $M_0 = M_2 = \frac{1}{2} M_1 = 5$ (blue) and $N = 50$, $M_0 = M_2 = \frac{1}{2} M_1 = 7$ (red).

The quantity $C_{N,M}$ against $N$, where $M_0 = M_2 = \frac{1}{2} M_1 = \lceil \sqrt{N} \rceil$. 
Numerical example II

Left: \( f(x) \). Right: Fourier series (black), generalized sampling with \( N = 100 \), \( M_0 = \ldots = M_4 = 13 \) (blue) and \( N = 200 \), \( M_0 = \ldots = M_4 = 18 \) (red).

The quantity \( C_{N,M} \) against \( N \), where \( M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{2} N} \rceil \).
Robustness I: noise

$M_0 = M_2 = \frac{1}{2} M_1 = \lceil \sqrt{N} \rceil$

$M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{2} N} \rceil$

Top row: $f(x)$. Bottom row: the error $\|f - f_{N,M}\|$ against $N$ with noise at amplitudes $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$. 
Robustness II: edge detection errors

Top row: $f(x)$. Bottom row: the error $\|f - f_{N,M}\|$ against $N$ with edge detection errors of magnitude $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$.

$M_0 = M_2 = \frac{1}{2} M_1 = \lceil \sqrt{N} \rceil$

$M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{2} N} \rceil$

▶ It can be shown that there is at worst linear drift in $M = \sqrt{N}$. 

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Robustness III: jitter

$M_0 = M_2 = \frac{1}{2} M_1 = \lceil \sqrt{N} \rceil$

Top row: $f(x)$.
Bottom row: the error $\|f - f_{N,M}\|$ against $N$ with jitter errors of magnitude $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$.

- It can be shown that the best achievable accuracy scales like $\mathcal{O}(\epsilon)$. 

$M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{2} N} \rceil$
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Motivation

In MRI one takes measurements

$$\{\mathcal{F} f(\omega_n)\}_{n=1}^N.$$ 

What types of sampling schemes $$\{\omega_n\}_{n=1}^N$$ are used in practice?

Equispaced:

1D

2D
Motivation

Jittered:

Logarithmic:
Motivation

We need robust reconstruction algorithms that can handle (potentially highly) nonuniform sampling strategies.

Problem: the sampling system

\[ \psi_n(x) = e^{2i\pi \omega_n \cdot x}, \quad n = 1, \ldots, N, \]

will not be orthonormal typically unless \( \{\omega_n\} \) are equispaced.
A system \( \{ \psi_n \}_{n \in \mathbb{N}} \) is a frame for a Hilbert space \( H \) if

- \( \{ \psi_n \}_{n \in \mathbb{N}} \) is dense in \( H \),
- there exist \( c_1, c_2 > 0 \) such that

\[
    c_1 \| g \|^2 \leq \sum_{n=1}^{\infty} |\langle g, \psi_n \rangle|^2 \leq c_2 \| g \|^2, \quad \forall g \in H.
\]

Note: the \textbf{frame operator} \( \mathcal{P} : H \to H, \mathcal{P}g = \sum_{n=1}^{\infty} \langle g, \psi_n \rangle \psi_n, \) is well-defined, bounded, self-adjoint, and we have

\[
    c_1 \| g \|^2 \leq \langle \mathcal{P}g, g \rangle \leq c_2 \| g \|^2.
\]
Generalized sampling with frames

Let

\[ \mathcal{P}_N g = \sum_{n=1}^{N} \langle g, \psi_n \rangle \psi_n, \]

and define

\[ \langle \mathcal{P}_N f, \phi_m \rangle = \langle \mathcal{P}_N f, \phi_m \rangle, \quad m = 1, \ldots, M. \]

Note:

- \( \mathcal{P}_N \to \mathcal{P} \) strongly on \( \mathbb{H} \).
- \( \langle \mathcal{P} \cdot, \cdot \rangle \) is an equivalent inner product on \( \mathbb{H} \).

Hence, generalized sampling works equally well for frames as it does for orthonormal bases.

- In particular, whenever \( \{\omega_n\} \) give rise to a Fourier frame then we may use generalized sampling.
The non-frame case

Jittered sampling often gives rise to a Fourier frame. However, log-sampling need not.

\[ \{ \omega_n \}_{n=1}^{N} \] may also depend on \( N \), i.e. \( \{ \omega_{n,N} \}_{n=1}^{N} \).

- In general, \( \{ \omega_n \}_{n=1}^{N} \) may also depend on \( N \), i.e. \( \{ \omega_{n,N} \}_{n=1}^{N} \).
Generalized sampling for (highly) nonuniform samples

Define the operator

\[ P_N g(x) = \sum_{n=1}^{N} \mu_{n,N} \mathcal{F} g(\omega_{n,N}) e^{2i\pi \omega_{n,N} x}. \]

- \( \mu_{n,N} = (\omega_{n+1,N} - \omega_{n,N}) \) is a density compensation factor.
- Local clustering of \( \{\omega_{n,N}\} \) is compensated by \( \mu_{n,N} \).

Work in progress: what conditions on \( \{\omega_{n,N}\}_{n=1}^{N} \) ensure that generalized sampling works for this (or a similar) choice of operator?

- Partial result: if \( \delta^N := \max |\omega_{n+1,N} - \omega_{n,N}| \leq \delta < 1, \forall N \in \mathbb{N} \), then generalized sampling works with \( P_N \) as above.
Numerical example I

Generalized sampling with $N = 20$, $M_0 = M_2 = \frac{1}{2} M_1 = 5$ (black), $N = 40$, $M_0 = M_2 = \frac{1}{2} M_1 = 4$ (blue) and $N = 80$, $M_0 = M_2 = \frac{1}{2} M_1 = 7$ (red).

The quantity $C_{N,M}$ against $N$, where $M_0 = M_2 = \frac{1}{2} M_1 = \lceil \frac{1}{2} \sqrt{N} \rceil$. 
Numerical example II

Generalized sampling with $N = 100$, $M_0 = \ldots = M_4 = 9$ (black), $N = 200$, $M_0 = \ldots = M_4 = 13$ (blue) and $N = 400$, $M_0 = \ldots = M_4 = 18$ (red).

The quantity $C_{N,M}$ against $N$, where $M_0 = \ldots = M_4 = \lceil \sqrt{\frac{3}{4}N} \rceil$. 
Robustness

Noise

Jitter

Edge detection
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Infinite-dimensional compressed sensing

Suppose $f$ is actually sparse in $\{\phi_n\}$, i.e.

$$|\{n : \alpha_n \neq 0\}| = k \ll \infty.$$

**Question:** can we recover $f$ exactly using $O(k)$ measurements?

**Answer:** yes! By combining generalized sampling ideas with existing tools from finite-dimensional compressed sensing.

- Leads to an important generalization of compressed sensing to infinite-dimensional signal models.
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- Leads to an important generalization of compressed sensing to infinite-dimensional signal models.
Theorem (BA,Hansen)

Given

\[ f = \sum_{n=1}^{\infty} \alpha_n \phi_n, \quad \Delta = \{ m : \alpha_n \neq 0 \}, \]

suppose that \( \Delta \subset \{1, \ldots, M\} \) for some \( M \in \mathbb{N} \). Let \( \epsilon > 0 \) be arbitrary. Then, there exists an integer \( N \in \mathbb{N} \) depending on \( M \) and \( |\Delta| \) only such that the following holds: if \( \Omega \subset \{1, \ldots, N\} \), \( |\Omega| = K \), is chosen uniformly at random, then, with probability greater than \( 1 - \epsilon \), \( f \) can be recovered exactly from the samples \( \{\hat{f}_m : m \in \Omega\} \) given that \( K \) is proportional to

\[ |\Delta| \cdot \log(\epsilon^{-1} + 1) \cdot \log(NM\sqrt{|\Delta|}). \]
Numerical example

Let $|\{n : \alpha_n \neq 0\}| = 25$, $\{\phi_n\}$ be Haar wavelets and set

$$f(x) = \sum_{n=1}^{200} \alpha_n \phi_n(x) + \chi_{[\frac{1}{2}, \frac{9}{16}]}(x) \cos 2\pi x, \quad x \in [0, 1].$$

<table>
<thead>
<tr>
<th>$M$</th>
<th>(a)</th>
<th>(b)</th>
<th>(c) (avg. 20 trials)</th>
</tr>
</thead>
<tbody>
<tr>
<td>601</td>
<td>1.43e0</td>
<td>4.74e-5</td>
<td>4.73e-5 ($m = 230$)</td>
</tr>
<tr>
<td>1201</td>
<td>8.5e-1</td>
<td>2.36e-5</td>
<td>2.38e-5 ($m = 460$)</td>
</tr>
</tbody>
</table>

Error for (a) the partial Fourier series, (b) generalized sampling and (c) generalized sampling with compressed sensing.
Generalized sampling


Generalized sampling with compressed sensing