

Chapter 3

Pure Integration Techniques

Design is not making beauty, beauty emerges from selection, affinities, integration, love.

—Louis Kahn

FUNDAMENTAL THEOREM OF CALCULUS

$$\begin{array}{ll} \text{(ODE)} & u' = f(t) \quad \iff \quad u(t) = \int f(t) dt \\ \text{(IVP)} & \begin{cases} u' = f(t), \\ u(0) = u_0 \end{cases} \quad \iff \quad u(t) = \int_0^t f(\tau) d\tau + u_0 \end{array}$$

For integrals which cannot readily be determined explicitly, a number of techniques can usually be used to transform them into simpler ones.

3.1 Simple integrals

- $\int t^n dt = \frac{1}{n+1} t^{n+1} + C \quad (n \neq -1)$
- $\int \frac{dt}{t} = \ln |t| + C$
- $\int \cos(nt) dt = \frac{1}{n} \sin(nt) + C \quad (n \neq 0)$
- $\int \sin(nt) dt = -\frac{1}{n} \cos(nt) + C \quad (n \neq 0)$
- $\int e^{at} dt = \frac{1}{a} e^{at} + C \quad (a \neq 0)$

Algebraic relations/formulas can sometimes be used to reduce an integral to a simple one.

Example 3.1

$$-4 \int \sin t \cos t dt \xrightarrow{(1.4c)} -2 \int \sin 2t dt \xrightarrow{\text{integrate}} \cos(2t) + C$$

■

3.2 Substitution

Substitution transforms an integral into one which can be evaluated simply.

Example 3.2

$$\begin{array}{ccc} \int \frac{\ln t}{t} dt & & \frac{1}{2}(\ln t)^2 + C \\ \begin{array}{c} t = e^\tau \\ dt = e^\tau d\tau \end{array} \downarrow & & \uparrow \tau = \ln t \\ \int \tau d\tau & \xrightarrow{\text{integrate}} & \frac{1}{2}\tau^2 + C \end{array}$$

■

Note that both t as a function of τ and τ as function of t are use in Example 3.2. Sometimes only τ in terms of t is needed, provided all terms in t can be converted to terms in τ during the substitution.

Example 3.3

$$\begin{array}{ccc} -4 \int \sin t \cos t dt & & -2 \sin^2 t + C \\ \begin{array}{c} \sin t = \tau \\ \cos t dt = d\tau \end{array} \downarrow & & \uparrow \tau = \sin t \\ -4 \int \tau d\tau & \xrightarrow{\text{integrate}} & -2\tau^2 + C \end{array}$$

The result looks different from that of Example 3.1. However the two functions $\cos(2t)$ and $-2 \sin^2 t$ only differ by a constant in view of (1.5b). ■

3.3 Integration by parts

The integration by parts formula

$$\int f(t)g'(t)dt = f(t)g(t) - \int f'(t)g(t)dt \quad (3.1)$$

is simply a restatement of the differentiation product rule (see Table page 5) combined with the fundamental Theorem of Calculus page 21 (move the second integral in (3.1) to the left, then differentiate both sides). This formula is useful whenever

1. the integrand is a product of two functions, one which can easily be differentiated ($f(t)$), the other one which can be simply integrated ($g'(t)$), AND

2. the resulting integral on the RHS of (3.1) is simpler than that on the LHS of (3.1).

The formula is sometimes written

$$\int f dg = fg - \int gdf. \quad (3.2)$$

Example 3.4

$$\int \ln t dt \xrightarrow{\begin{array}{l} f = \ln t \Rightarrow df = dt/t \\ dg = dt \Rightarrow g = t \end{array}} t \ln t - \int t \frac{dt}{t} = t \ln t - t + C$$

■

Example 3.5

$$\int e^t \sin t dt \xrightarrow{\begin{array}{l} f = \sin t \Rightarrow df = \cos t dt \\ dg = e^t dt \Rightarrow g = e^t \end{array}} e^t \sin t - \int e^t \cos t dt$$

$$\xrightarrow{\begin{array}{l} f = \cos t \Rightarrow df = -\sin t dt \\ dg = e^t dt \Rightarrow g = e^t \end{array}} e^t \sin t - \left(e^t \cos t + \int e^t \sin t dt \right)$$

so that $2 \int e^t \sin t dt = e^t(\sin t - \cos t) + C$ (the constant of integration has to be added). See (3.7b) for a more general formula. ■

3.4 Partial fraction decomposition

The Partial Fraction Decomposition (PFD) rewrites a rational function $F(s) = \frac{P(s)}{Q(s)}$ with $\text{degree}(P) < \text{degree}(Q)$ as a *sum of simpler rational functions*. The process involves three phases:

1. Factoring Q . $Q(s)$ is written as a product of linear terms and quadratic terms with no real root (irreducible). Linear terms may be repeated and appear as $(s - \alpha)$, $(s - \alpha)^2$, ... Irreducible quadratic terms should be written in the form $(s - \alpha)^2 + \beta^2$, $((s - \alpha)^2 + \beta^2)^2$, ... First identify α by *completing the square*, see Section 1.5.
2. Set-up. Write $F(s)$ as an a-priori sum of terms involving unknown coefficients. Certain rules must be applied in order to obtain a correct set-up:

A factor like this in $Q(s)$...	yields a contribution like this in the PFD ...
$(s - \alpha)$	$\frac{a}{s - \alpha}$
$(s - \alpha)^2$	$\frac{a}{s - \alpha} + \frac{b}{(s - \alpha)^2}$
$(s - \alpha)^2 + \beta^2$	$\frac{a(s - \alpha) + b}{(s - \alpha)^2 + \beta^2}$
$((s - \alpha)^2 + \beta^2)^2$	$\frac{a(s - \alpha) + b}{(s - \alpha)^2 + \beta^2} + \frac{c(s - \alpha) + d}{((s - \alpha)^2 + \beta^2)^2}$

3. Determine unknown coefficients. The set-up phase leads $\frac{P(s)}{Q(s)} = \text{PFD}$, i.e., $P(s) = Q(s) \text{PFD}$. Simplify and either substitute appropriate values for s (APPROACH 1) or expand and compare powers of s (APPROACH 2). This yields a system of linear equations for the unknown coefficients.

Example 3.6 Determine $\int F(s) ds$ with $F(s) = \frac{1}{s^2 + 3s + 2}$:

- Factor denominator: $s^2 + 3s + 2 = (s + 1)(s + 2)$.
- Set-up: $\frac{1}{(s + 1)(s + 2)} = \frac{a}{s + 1} + \frac{b}{s + 2}$.
- Multiply by common denominator and simplify: $1 = a(s + 2) + b(s + 1)$.

APPROACH 1: substitute values

APPROACH 2: compare powers of s :

$$\begin{array}{ll} s = -1 \Rightarrow 1 = a & s^0 : 1 = 2a + b \Rightarrow a = 1 \\ s = -2 \Rightarrow 1 = -b & s^1 : 0 = a + b \Rightarrow b = -1 \end{array}$$

Thus $F(s) = \frac{1}{s + 1} - \frac{1}{s + 2}$ and $\int F(s) ds = \ln |s + 1| - \ln |s + 2| + C = \ln \left| \frac{s + 1}{s + 2} \right| + C$.

■

Example 3.7 Determine $\int F(s) ds$ with $F(s) = \frac{s + 2}{(s^2 + 2s + 1)(s^2 + 2s + 2)}$:

- Factor denominator: $(s^2 + 2s + 1)(s^2 + 2s + 2) = (s + 1)^2((s + 1)^2 + 1)$.
- Set-up: $F(s) = \frac{a}{s + 1} + \frac{b}{(s + 1)^2} + \frac{c(s + 1) + d}{(s + 1)^2 + 1}$.
- Multiply by common denominator and simplify:

$$s + 2 = a(s + 1)((s + 1)^2 + 1) + b((s + 1)^2 + 1) + (c(s + 1) + d)(s + 1)^2.$$

APPROACH 1: substitute values

$$\begin{array}{ll} s = -1 \Rightarrow 1 = b & \\ s = 0 \Rightarrow 2 = 2a + 2b + c + d & \\ s = -2 \Rightarrow 0 = -2a + 2b - c + d & \\ s = 1 \Rightarrow 3 = 10a + 5b + 8c + 4d & \end{array} \quad (3.3)$$

Pick values until you get as many equations as unknowns. The choice $s = -1$ is obvious (most terms vanish), while other choices are somewhat arbitrary.

APPROACH 2: expand in powers of s , or, better here, $(s + 1) = z$ ($s = z - 1$),

$$z + 1 = az(z^2 + 1) + b(z^2 + 1) + (cz + d)z^2 = (a + c)z^3 + (b + d)z^2 + az + b$$

and compare powers of z :

$$\begin{array}{ll} z^0 : 1 = b & a = 1 \\ z^1 : 1 = a & b = 1 \\ z^2 : 0 = b + d & \Rightarrow c = -1 \\ z^3 : 0 = a + c & d = -1 \end{array} \quad (3.4)$$

We verify that (3.4) satisfies the system (3.3). Thus

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}. \quad (3.5)$$

There is a strong incentive in leaving the last two terms of (3.5) separate. Indeed,

$$\int F(s) ds = \boxed{\int \frac{ds}{s+1}} + \boxed{\int \frac{ds}{(s+1)^2}} - \boxed{\int \frac{s+1}{(s+1)^2+1} ds} - \boxed{\int \frac{ds}{(s+1)^2+1}} + C$$

$$\begin{array}{ccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ln|s+1| & & -\frac{1}{s+1} & & \frac{1}{2} \ln((s+1)^2+1) & & \arctan(s+1) \quad \blacksquare \end{array}$$

APPROACH 1 works best when all factors in the denominator are of the form $(s - \alpha)$. APPROACH 2 is safer (the set-up is incorrect if no solution is found) but may lead to systems which are more difficult to solve. A combination of both strategies may be used. A matrix elimination technique for solving systems is also described in Chapter ...

3.5 Trigonometric integrals

Many trigonometric integrals require a combination of algebraic manipulations, substitution(s), and/or a PFD.

Example 3.8

$$\begin{array}{l} \int \frac{dt}{\cos t} = \int \frac{\cos t}{\cos^2 t} dt \qquad \frac{1}{2} \ln \frac{1 + \sin t}{1 - \sin t} + C \\ \quad \downarrow \begin{array}{l} \sin t = \tau \\ \cos t dt = d\tau \end{array} \qquad \qquad \qquad \uparrow \tau = \sin t \\ \int \frac{d\tau}{1 - \tau^2} \xrightarrow{PFD} \int \left(\frac{1}{1 + \tau} + \frac{1}{1 - \tau} \right) \frac{d\tau}{2} \xrightarrow{\text{integrate}} \frac{1}{2} \ln \left| \frac{1 + \tau}{1 - \tau} \right| + C \end{array}$$

Since $1 + \sin t = 1 + \cos(\frac{\pi}{2} - t) = 2 \cos^2(\frac{\pi}{4} - \frac{t}{2})$ and $1 - \sin t = 1 - \cos(\frac{\pi}{2} - t) = 2 \sin^2(\frac{\pi}{4} - \frac{t}{2})$ the result can also be written in the alternate form

$$\frac{1}{2} \ln \frac{1 + \sin t}{1 - \sin t} = \frac{1}{2} \ln \cot \left(\frac{\pi}{4} - \frac{t}{2} \right)^2 = - \ln \left| \tan \left(\frac{\pi}{4} - \frac{t}{2} \right) \right|.$$

This is the form one usually finds in integral Tables. ■

When every “obvious” simple substitution fails, consider the substitution

$$\tau = \tan \frac{t}{2} \Rightarrow \sin t = \frac{2\tau}{1 + \tau^2}, \quad \cos t = \frac{1 - \tau^2}{1 + \tau^2}, \quad dt = \frac{2d\tau}{1 + \tau^2}. \quad (3.6)$$

Example 3.9

$$\begin{aligned} \int \frac{dt}{\cos t} & \qquad \qquad \qquad \ln \left| \frac{1 + \tan \frac{t}{2}}{1 - \tan \frac{t}{2}} \right| + C \\ \downarrow \tau = \tan \frac{t}{2} & \qquad \qquad \qquad \uparrow \tau = \tan \frac{t}{2} \\ \int \frac{2d\tau}{1 - \tau^2} \xrightarrow{PFD} \int \left(\frac{1}{1 + \tau} + \frac{1}{1 - \tau} \right) d\tau \xrightarrow{\text{integrate}} \ln \left| \frac{1 + \tau}{1 - \tau} \right| + C \end{aligned}$$

The verification that this result is identical to the one obtained in Example 3.8 is left as Problem 1.55. ■

Typical mistakes and useful tips

- [mistakes] Typical integration mistakes

wrong	correct	explanation
$\int \frac{dx}{x} = \ln(x) + C$ $\int \frac{dx}{x^2 + 1} = \ln(x^2 + 1) + C$	$\int \frac{dx}{x} = \ln x + C$ $\int \frac{dx}{x^2 + 1} = \arctan(x) + C$	errors with ln
$\int \ln(x) dx = \frac{1}{x} + C$	$\int \ln(x) dx = x \ln(x) - x + C$	do not differentiate
$\int \frac{dx}{x^2} = -\frac{1}{x}$	$\int \frac{dx}{x^2} = -\frac{1}{x} + C$	integration constant

- [tip] The formulas

$$\int e^{at} \cos(bt) dt = e^{at} \left(\frac{a}{a^2 + b^2} \cos(bt) + \frac{b}{a^2 + b^2} \sin(bt) \right) + C \quad (3.7a)$$

$$\int e^{at} \sin(bt) dt = e^{at} \left(\frac{-b}{a^2 + b^2} \cos(bt) + \frac{a}{a^2 + b^2} \sin(bt) \right) + C \quad (3.7b)$$

will prove useful in this course and are worth memorizing. To verify the identities simply differentiate both sides. In case you forget them they can be obtained in a number of ways:

1. [undetermined coefficients] Simply remember that both integrals are of the form $e^{at}(\alpha \cos(bt) + \beta \sin(bt)) + C$ with α and β to be determined by differentiating and comparing $e^{at} \cos(bt)$ and $e^{at} \sin(bt)$ terms.
 2. [double integration by parts] Integrate $\int e^{at} \cos(bt) dt$ by parts (integrate e^{at} , differentiate $\cos(bt)$) and get a term $\int e^{at} \sin(bt) dt$. Another integration by parts (integrate e^{at} , differentiate $\sin(bt)$) yields an equation for $\int e^{at} \cos(bt) dt$. See Example 3.5.
 3. [using Euler's formula] Express $\sin(bt)$ and $\cos(bt)$ in terms of $e^{\pm ibt}$. Integrate the resulting exponentials and simplify the complex expression obtained using Euler's formula again (the result is a real function!).
- **[tip]** Be aware of alternate strategies/simplifications. A striking example of a needlessly complicated solution is the following:

$$\begin{aligned}
 & \int \frac{dt}{\sqrt{t} + \sqrt{t+1}} \\
 & \quad \downarrow t = \tau^2 \text{ (standard with integrals involving } \sqrt{t}\text{)} \\
 & \int \frac{2\tau d\tau}{\tau + \sqrt{\tau^2 + 1}} \\
 & \quad \downarrow \tau = \tan \theta \text{ (standard with integrals involving } \sqrt{t^2 + 1}\text{)} \\
 & \int \frac{2 \tan \theta \sec^2 \theta}{\tan \theta + \sec \theta} d\theta = \int \frac{2 \sin \theta}{\cos^2 \theta (1 + \sin \theta)} d\theta \\
 & \quad v = \tan \frac{\theta}{2} \text{ (see (3.6))} \downarrow \\
 & \int \frac{8v(1+v^2)dv}{(1-v^2)^2(1+v)^2} \\
 \text{PDF } & \frac{a}{1-v} + \frac{b}{(1-v)^2} + \frac{c}{1+v} + \frac{d}{(1+v)^2} + \frac{e}{(1+v)^3} + \frac{f}{(1+v)^4} \downarrow \\
 & \int \left(\frac{1}{(1-v)^2} - \frac{1}{(1+v)^2} + \frac{4}{(1+v)^3} - \frac{4}{(1+v)^4} \right) dv \\
 & \quad \text{integrate} \downarrow \\
 & \frac{1}{1-v} + \frac{1}{1+v} - \frac{2}{(1+v)^2} + \frac{4/3}{(1+v)^3} + C \quad (3.8)
 \end{aligned}$$

with $v = \tan \frac{\theta}{2} = \tan \left(\frac{1}{2} \arctan \tau \right) = \tan \left(\frac{1}{2} \arctan \sqrt{t} \right) = v(t)$.
 Had we realized instead that

$$\frac{1}{\sqrt{t} + \sqrt{t+1}} = \frac{\sqrt{t+1} - \sqrt{t}}{(\sqrt{t} + \sqrt{t+1})(\sqrt{t+1} - \sqrt{t})} = \sqrt{t+1} - \sqrt{t}$$

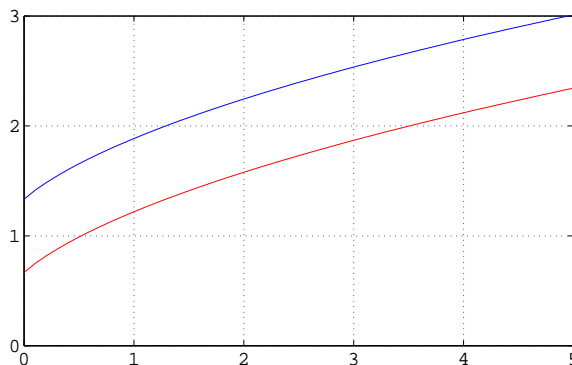


Figure 3.1. The functions (3.8) (with $C = 0$, top) and (3.9) (with $C = 0$, bottom) differ by a constant ($\frac{2}{3}$).

we would have easily obtained

$$\int \frac{dt}{\sqrt{t} + \sqrt{t+1}} = \int (\sqrt{t+1} - \sqrt{t}) dt = \frac{2}{3} \left((t+1)^{\frac{3}{2}} - t^{\frac{3}{2}} \right) + C. \quad (3.9)$$

Both (3.8) and (3.9) are correct. The fact that these functions only differ by a constant is not at all obvious algebraically but can be verified numerically, see Figure 3.1.

Different integration strategies may lead to different but equally legitimate forms of a result, some better than others for the purpose we have in mind. This is especially true when dealing with trigonometric expressions. This chapter gives you a few tools and ideas to get to at least one of these forms.

Exercises

In Problems 3.1-3.116 determine the given definite or indefinite integral.

3.1. $\int (t^{2/5} + 4\sqrt{t} - 7 \sin t) dt$

3.6. $\int \frac{2t - 9}{t^2 - 6t + 9} dt$

3.2. $\int t e^t dt$

3.7. $\int t^3 \ln(t) dt$

3.3. $\int \frac{\ln(5t)}{t} dt$

3.8. $\int \arctan(\sqrt{t}) dt$

3.4. $\int \left(\frac{t^4}{3} + \frac{8}{t} \right) dt$

3.9. $\int \frac{1}{\sqrt{t-e}} dt$

3.5. $\int e^{5t} dt$

3.10. $\int \frac{5y^2 + 3y + 8}{(y^2 + 1)(y + 1)} dy$

- 3.11. $\int \sin(t)\sqrt{3+2\cos(t)} dt$
- 3.12. $\int t^2 e^{t^3} dt$
- 3.13. $\int \frac{(u^2+1)^2}{\sqrt{u}} du$
- 3.14. $\int t \sin(2t) dt$
- 3.15. $\int (t+1)e^t dt$
- 3.16. $\int \frac{e^{1/t}}{t^2} dt$
- 3.17. $\int (t\sqrt[3]{t} - 6\sin(t)) dt$
- 3.18. $\int \left(\frac{t^5}{2} - \frac{7}{\sqrt{t}}\right) dt$
- 3.19. $\int_0^{\pi/4} \sqrt{\tan(t)} \sec^2(t) dt$
- 3.20. $\int_1^4 \frac{\ln(4t)}{t} dt$
- 3.21. $\int_{-2}^{-1} |t| dt$
- 3.22. $\int e^t \ln(1+e^t) dt$
- 3.23. $\int t^4 \ln(t) dt$
- 3.24. $\int \sin(t) \cos(t) e^{\sin(t)} dt$
- 3.25. $\int \arctan(t) dt$
- 3.26. $\int \frac{u}{\sqrt{1-u^2}} du$
- 3.27. $\int \frac{4w^2 - w + 2}{(w^2 + 6)(w + 2)} dw$
- 3.28. $\int \sqrt{t} \ln(t) dt$
- 3.29. $\int \frac{e^t}{(e^t + 2)^2} dt$
- 3.30. $\int t e^t dt$
- 3.31. $\int \frac{y+1}{\sqrt{y-1}} dy$
- 3.32. $\int \frac{4t+21}{t^2+9t+18} dt$
- 3.33. $\int (t^{4/3} - 2\sqrt{t} + 3\cos(2t)) dt$
- 3.34. $\int \left(\frac{t^4}{3} - \frac{6}{t^2}\right) dt$
- 3.35. $\int \cos(2t) \cos(\sin(2t)) dt$
- 3.36. $\int t\sqrt{t-1} dt$
- 3.37. $\int \frac{2e^t + 4t}{e^t + t^2} dt$
- 3.38. $\int t^6 \ln(2t) dt$
- 3.39. $\int_1^2 \frac{e^{3t}}{e^{3t} + 1} dt$
- 3.40. $\int \frac{3t+1}{(t^2+1)(t+1)} dt$
- 3.41. $\int \frac{t}{1+t^4} dt$
- 3.42. $\int \frac{\cos(2t)}{\sin(2t)+1} dt$
- 3.43. $\int t \cos(3t) dt$
- 3.44. $\int \frac{t-7}{t^2+t-6} dt$
- 3.45. $\int \sqrt[3]{t} \ln(t) dt$
- 3.46. $\int \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$
- 3.47. $\int \frac{t}{t-1} dt$
- 3.48. $\int \frac{1}{\sqrt{2t-3}} dt$
- 3.49. $\int \left(t^2 - \frac{1}{4t}\right) dt$
- 3.50. $\int (\sqrt[3]{t} + \sqrt{t} + 5\cos(t)) dt$
- 3.51. $\int \frac{t^2 + 4t + 5}{(t+1)^2(t+2)} dt$
- 3.52. $\int \frac{t^5}{\sqrt{1-t^2}} dt$

- 3.53. $\int (e^t + 2t)(e^t + t^2)^8 dt$
- 3.54. $\int \frac{e^{\ln(t)}}{t} dt$
- 3.55. $\int t^2 \ln(t) dt$
- 3.56. $\int \frac{e^{2t}}{e^{2t} + 1} dt$
- 3.57. $\int w^4 e^{w^5} dw$
- 3.58. $\int \frac{2t^2 + 14t + 21}{(1 - 2t)(t^2 + 3t + 3)} dt$
- 3.59. $\int_{-1}^1 \arcsin(t) dt$
- 3.60. $\int \sqrt{t}(1 + \ln(t)) dt$
- 3.61. $\int \frac{3t^2 + 18t - 10}{(t + 4)(t^2 + 1)} dt$
- 3.62. $\int \frac{1}{\sqrt{3t + \pi}} dt$
- 3.63. $\int \frac{(\ln(2t))^2}{t} dt$
- 3.64. $\int \left(t^3 + \frac{1}{t} \right) dt$
- 3.65. $\int \frac{1}{t^{1/4}} + 3 \sin(t) + \sqrt{t} dt$
- 3.66. $\int \frac{5}{e^{2t} + 1} dt$
- 3.67. $\int \sqrt{t + \sin(t)}(1 + \cos(t)) dt$
- 3.68. $\int \frac{\sin(\ln 2t)}{t} dt$
- 3.69. $\int_1^1 e^{-t^2} dt$
- 3.70. $\int \frac{3t + 1}{t^2 - 2t + 1} dt$
- 3.71. $\int \arccos(t) dt$
- 3.72. $\int_{-2}^2 \sin(t^3) dt$
- 3.73. $\int w^4 \cos(w^5) dw$
- 3.74. $\int \ln(t + 2) dt$
- 3.75. $\int (2t + 1)\sqrt{t + 2} dt$
- 3.76. $\int \frac{2t^2 - 3t + 2}{t(t - 1)^2} dt$
- 3.77. $\int \frac{t - 1}{\sqrt{t}} dt$
- 3.78. $\int \frac{1 + e^{2t}}{e^t} dt$
- 3.79. $\int \frac{t^2}{t + 1} dt$
- 3.80. $\int \frac{dt}{t(\ln(t))^2}$
- 3.81. $\int \frac{1}{\sqrt{1 - 4t^2}} dt$
- 3.82. $\int_0^{\pi/4} \sec^2 t dt$
- 3.83. $\int \sin^5 t dt$
- 3.84. $\int \tan t dt$
- 3.85. $\int_0^{\pi/3} \sec^2 \beta \tan \beta d\beta$
- 3.86. $\int_0^{\pi/3} \sin^3 t dt$
- 3.87. $\int \frac{\sec \alpha \tan \alpha}{(1 + \sec \alpha)^2} d\alpha$
- 3.88. $\int \sec t \tan t dt$
- 3.89. $\int \sec^5 \theta \tan \theta d\theta$
- 3.90. $\int e^{\sqrt{t}} dt$
- 3.91. $\int \frac{1}{1 + \sqrt{t}} dt$
- 3.92. $\int \cos^2 t dt$
- 3.93. $\int \sin^4 t dt$
- 3.94. $\int \frac{ds}{s(1 + (\ln s)^2)}$

- 3.95. $\int_0^1 \frac{1}{e^t} dt$
- 3.96. $\int \sin 5t \sin 2t dt$
- 3.97. $\int \sin^3 t \cos^2 t dt$
- 3.98. $\int \sqrt{4-t^2} dt$
- 3.99. $\int \frac{2t-2}{(t^2-2t+3)^5} dt$
- 3.100. $\int \frac{3y^2+y+1}{y^3+y} dy$
- 3.101. $\int_0^1 \frac{t^3}{\sqrt{1-t^2}} dt$
- 3.102. $\int \cos t \sin^3 t dt$
- 3.103. $\int \frac{1}{t^2-6t+10} dt$
- 3.104. $\int r\sqrt{r^2+4} dr$
- 3.105. $\int_0^{\pi/3} \frac{\sin t}{\cos^3 t} dt$
- 3.106. $\int_0^1 \sqrt{z}(z+\sqrt[3]{z}) dz$
- 3.107. $\int_0^2 \frac{2t}{(t-3)^2} dt$
- 3.108. $\int \frac{1}{2^t} dt$
- 3.109. $\int \sin 3t \cos t dt$
- 3.110. $\int \cos 4t \cos 2t dt$
- 3.111. $\int t^3 e^{t^2} dt$
- 3.112. $\int \tan^3 t \sec^3 t dt$
- 3.113. $\int \tan^2 t dt$
- 3.114. $\int \frac{1}{t^2\sqrt{t^2-9}} dt$
- 3.115. $\int \frac{t^3}{\sqrt{t^2+9}} dt$
- 3.116. $\int_1^2 \sqrt{2t-t^2} dt$